INDIAN STATISTICAL INSTITUTE

CHENNAI CENTRE

M.STAT FIRST YEAR 2014-15 SEMESTER-I

STATISTICAL INFERENCE

Final Examination

Total Marks : 60 Duration : 3hours

Answer any seven Questions

(1) a) Suppose X_1 and X_2 are independent and identically distributed observations from the pdf

$$f(x|\alpha) = \alpha\, x^{\alpha-1}\, e^{-x^{\alpha}}\ , x > 0,\ \alpha > 0.$$

Show that $\log X_1 / \log X_2$ is an ancillary statistic.

b) Let $(X_1, X_2, \ldots, X_n)$ $n > 2$ be a random sample from the exponential distribution on (a,∞) with scale parameter θ. Show that $\sum_{i=1}^{n}(X_i - X_{(1)})$ and $X_{(1)}$ are independent for any (a,θ), when $X_{(j)}$ in the jth order statistic.

(2) Let X be a single observation from a pdf with $e^{-x+\theta}\ \mathrm{I}_{(\theta,\infty)}(X)$, where $\theta > 0$ is an unknown parameter. Consider the estimate of

$$t = \begin{cases} j & \theta \in [j-1, j], \quad j = 1,2,3 \\ 4 & \theta > 3 \end{cases}$$

Under the loss L(i,j), 1$\leq$ i, j$\leq$ 4 given by the following matrix

$$\begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 2 \\ 1 & 2 & 0 & 2 \\ 3 & 3 & 3 & 0 \end{pmatrix}$$

When X=4, find the Bayes action w.r.t prior with a pdf $e^{-\theta}\ \mathrm{I}_{(0,\infty)}(\theta)$ (i.e) $f(\theta) = e^{-\theta}$, $\theta > 0$. Note that bayes action is one which minimizes the posterior expected loss.

(3) Let θ={0, 1} A={0, 1} and let the loss function be

$$\mathrm{L}(0,0) = \mathrm{L}(1,1) = 0,$$

$$\mathrm{L}(1,0) = \mathrm{L}(0,1) = 1$$

(The statistician tries to guess what nature has chosen. If he guesses correctly he loses nothing. If he guesses incorrectly he loses 1.)
Suppose the statistician observes the random variable X with the distribution

$$P(X = x|\theta) = 2^{-k} \text{ if } x = k - \theta \text{ for } \quad k = 1, 2, \ldots$$

(a) Describe the set of all non randomized decision rules.

(b) Plot the risk set S in the plane.

(c) Find the minimal complete class of decision rules.

(4) Show that $\overline{X}$ is admissible as an estimate of the mean of a normal distribution under the absolute error loss. (State your assumptions clearly.)

(5) Let X have a discrete distribution with probability mean function

$$f(x|\theta) = \begin{cases} \theta & \text{if } x = -1 \\ (1-\theta)^2\theta^x & \text{if } x = 0, 1, 2, \ldots \end{cases}$$

where $0 < \theta < 1$.
Show that X is a boundedly complete sufficient statistic for θ but X is not a complete sufficient statistic for θ.

(6) Suppose that a given decision problem is invariant under a finite group. If an invariant rule $\delta_o \in D$ is admissible within the class of invariant rules , show that it is admissible.

(7) Let $\theta = [0, 1]$ =A. (i.e) $0 \leq \theta \leq 1$. Let the loss function be

$$\mathrm{L}(\theta, a) = (1-\theta)a + \theta(1-a)$$

and let X have the binomial distribution B(n,θ). Show that loss function is convex and hence we can restrict our attention to nonrandomized rules.

(a) Show that this problem is invariant under the group of two elements generated by $g(x) = n - x$ and find $\bar{g}(\theta)$ and $\tilde{g}(a)$.

(b) Find the form of the nonrandomized invariant decision rules.

(c) Show that the invariant rule

$$d(x) = 0 \text{ for } x < n/2,\ d\left(\frac{n}{2}\right) = 1/2$$

$d(x) = 1$, for $x > n/2$ is best invariant decision rule.

(d) If $d(x) \equiv 1/2$ is a minimax rule. Compare .

(8) Show that if a minimal complete class exists, it consists of exactly admissible rules.

(9) Let X have a Cauchy distribution with unknown median θ and known semi-interquartile range β and let the loss function be

$$L(\theta, a) = \begin{cases} 0 & \text{if } |a-\theta| \leq C \\ 1 & \text{if } |a-\theta| > C. \end{cases}$$

where C is a given positive number. Find the best invariant estimate of θ based on X and note that it does not depend on β or C.

(10) Suppose that a given decision problem is invariant under a finite group $\mathfrak{T}$.

(a) If δ_o is Bayes with respect to and δ_o is invariant, there exists an invariant prior distribution τ_o with respect to which δ_o is Bayes.

(b) If δ is Bayes with respect to τ_o and τ_o is invariant, there exist an invariant rule δ_o which is Bayes with respect to τ_o.

(11) Let $X_1, \ldots, X_4$, be four observations from a bernoulli distribution with probability of success 'p'. Consider the problem of testing $H_o : p = 1/2$ against $H_1 : p = 3/4$

(a) Find the most powerful test of size $\alpha = 0.05$.

(b) If L(1/2, 1/2) = L(3/4, 3/4) =0 ; L(1/2, 3/4) =1 and L(3/4, 1/2) =2, find a minimax rule.

(c) If prior probabilites are $\pi(1/2) = 1/3$ and $\pi(3/4) = 2/3$, find the Bayes rule.