INDIAN STATISTICAL INSTITUTE
CHENNAI CENTRE
M.STAT First Year
2014-15 Semester I

Real Analysis
Final Examination

Total Points 100. Duration: 3 hours

Answer Question no. 1 and any 5 from the rest.

1.(a) Let (X, d) be a metric space. For any $A, B \subseteq X$, define $d_p(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. Does d_p define a metric on the power set $\mathcal{P}(X)$? Justify your answer.

(b) Give an example (giving justifications) of a set in $\mathbb{R}^2$ which is neither open nor closed.

(c) Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at $c \in \mathbb{R}$ but not continuous on any open interval containing c? Justify your answer.

(d) Let f be a bijective, continuous function defined on a compact subset of $\mathbb{R}$. Must f^{-1} be continuous on its domain of definition? Justify your answer.

(e) Let $A = \{(x, y, z) \in \mathbb{R}^3 : x = y = z\}$. Is A open in $\mathbb{R}^3$? Justify your answer.

$5 \times 3 = 15$

2.(a) Determine all the points x in $\mathbb{R}$ for which the limit, $\lim_{n\to\infty} x^n/(1 + x^{2n})$ exists.

(b) Does $\sum_{n=1}^{\infty}(1/n)\sin(4n)$ converge? Justify your answer.

(c) Find a point of local extremum (in case it exists) for the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x, y) = 3x^5 - 7xy^4$. $6 + 6 + 5 = 17$

3.(a) Every open subset of $\mathbb{R}$ is the union of a countable collection of closed sets: Prove or disprove.

(b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that $f(x) = x.f(1)$ and $f(x) = x.f'(0)$ for all $x \in \mathbb{R}$.

$9 + 8 = 17$

4.(a) Let X be a metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a convergent sequence in X with limit x_0. Show that the set $\{x_0, x_1, x_2, \ldots\}$ is compact.

(b) Let $\{f_n\}$ be a sequence of non-decreasing functions defined from $[0, 1]$ to $[0, 1]$. Suppose that $f_n(x)$ converges to $f(x)$ pointwise on $[0, 1]$, where f is a continuous function. Does f_n converge uniformly to f on $[0, 1]$? Justify your answer.

(c) Evaluate the double integral

$$I = \iint\limits_{\mathbf{R}} \frac{\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}}{\sqrt{1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}}} dxdy,$$

where $\mathbf{R}$ is the region in the first quadrant bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$4 + 8 + 5 = 17$

5.(a) Prove that the intersection of two open sets in $\mathbb{R}$ is compact iff they are disjoint.

(b) Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Do the partial derivatives and directional derivatives exist at $(0,0)$? Is f differentiable at $(0,0)$? Justify your answer in each case.

(c) Let $f : [0,1] \to \mathbb{R}$ defined by,

$$f(x) = \begin{cases} x.\cos(\pi/2x) & \text{if } 0 < x \leq 1; \\ 0 & \text{if } x = 0. \end{cases}$$

Is f rectifiable? Justify your answer. $6 + 5 + 6 = 17$

6.(a) Let X be a set, (Y, d) be a metric space, and $f : X \to Y$. Define $d_X : X \times X \to \mathbb{R}$ by $d_X(x,y) = d(f(x), f(y))$. Verify whether d_X defines a metric on X. If not, can you impose some restrictions on f to make it a metric? Give reasons for your answers.

(b) Evaluate the triple integral

$$I = \iiint\limits_{\mathbf{D}} \frac{z}{(x^2 + y^2 + z^2)^{3/2}} dxdydz,$$

where $\mathbf{D} = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 16, z \geq 0\}$.

(c) Let (X, d) be a metric space, and $a \in X$. Define a map $f : X \to \mathbb{R}$ by, $f(x) = d(x,a)$. Is f continuous on X? Justify your answer. $7 + 6 + 4 = 17$

7.(a) Let X be the set of all sequences over $\{0,1\}$, with finitely many 1's. Is X countable? Justify your answer.

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function and $f : \mathbb{R} \to \mathbb{R}$ be defined by,

$$f(x) = \int_0^x (x - t) g(t) dt.$$

Show that f'' exists on $\mathbb{R}$ and $f''(x) = g(x)$ for all $x \in \mathbb{R}$.

(c) Maximize the function $f(x,y,z) = x - 3y + 2z$ subject to the constraint $x^2 + y^2 + z^2 = 7$.

$7 + 5 + 5 = 17$