INDIAN STATISTICAL INSTITUTE
CHENNAI CENTRE
M.STAT First Year
2014-15 Semester I

Linear Models
Final Examination

Total Points 50.   Duration: 1.5 hours

1. (a) The quantiles of normal distribution are

$$q_1 = 18.7189,$$
$$q_2 = 21.6681,$$
$$q_3 = 25.5026.$$

Estimate by method of least squares, the mean and standard deviation of the distribution. (5 points)

(b) Discuss how a projection matrix P can be used to obtain the Best Linear Unbiased Esimator (BLUE) for a linear parametric function $\lambda\beta$. (5 points)

2. (a) Prove or disprove if every Best Linear Unbiased Estimator (BLUE) is expressed in terms of observed y as $a^T y$ then a is linear combination of X. (2.5 points)

(b) Prove or disprove if every linear function $a^T y$ where a is a linear combination of X then $a^T y$ is the BLUE of its expected value. (2.5 points)

(c) Prove or disprove the coefficient vector of any BLUE is orthogonal to the coefficient vector of y belonging to the error space. (3 points)

(d) If P is a projection matrix then show that P is also an idempotent matrix. (2 points)

3. Suppose

$$SSE = (y - X\beta)^T (y - X\beta)$$

is the sum of squares of the model

$$y = X\beta + \epsilon$$

such that $E(\epsilon) = 0$ and $Var(\epsilon) = \sigma^2 \mathbf{I}_n$ where y is observed, $X^{n \times p}$ is known, $rank(X) = r \leq p$ and β, σ^2 are unknown.

(a) Show that $MSE = \frac{SSE}{n-r}$, i.e., mean square error of the model is an unbiased estimator of σ^2 (4 points)

(b) If $\epsilon \sim N(0, \sigma^2)$ then what is the sampling distribution of $\hat{\beta}$ where $\hat{\beta}$ is the solutions of the normal equations? (2 points)

(c) What is the sampling distribution of $\lambda\hat{\beta}$ for $\lambda\beta$? (2 points)

(d) Are $\hat{\beta}$ and $\lambda\hat{\beta}$ always unbiased estimator and are these sampling distribution always non-singular? (2 points)

4. Suppose

$$SSE = (y - X\beta)^T(y - X\beta)$$

is the sum of squares of the model

$$y = X\beta + \epsilon$$

such that $\epsilon \sim N(0, \sigma^2.I_n)$ where y is observed, $X^{n \times p}$ is known, $rank(X) = r$ and β, σ^2 are unknown.

(a) Show that $SSE \sim \sigma^2 \chi^2_{n-r}$ (4 points)

(b) To test the following hypothesis

$$H_0 : \quad \Lambda\beta = d \quad vs. \quad H_A : \quad Not \ H_0$$

where

$$\Lambda = \begin{bmatrix} \lambda^T_{(1)} \\ \lambda^T_{(2)} \\ \vdots \\ \lambda^T_{(m)} \end{bmatrix}^T$$

is a $m \times p$ matrix and

$$d = (d_1, d_2, ..., d_m)$$

is a colum vector. Provide the test-statistic and corresponding test for the null hypothesis. (4 points)

(c) Discuss how we can evaluate the power function of the test. (2 points)

5. (a) State the Cochran's Theorem and discuss the motivation of the proof for two compotents only (5 points)

(b) Suppose $X_i \sim N(0, \sigma^2)$ for all $i = 1, 2, ..., n$. Discuss the distribution of $\sum_{i=1}^{n} X_i^2$, $\sum_{i=1}^{n}(x_i - \bar{x})^2$ and $(\frac{\sum_i x_i}{n})^2$. (3 points)

(c) Discuss if the distribution of $\sum_{i=1}^{n}(x_i - \bar{x})^2$ and $(\frac{\sum_i x_i}{n})^2$ are independent! (2 points)