Probability Theory
Semestral Examination
Duration: 180 minutes

(1) r distinguishable objects are placed in n cells. Let p_k be the probability that exactly k cells are empty. Show that

(a)

$$p_k = \sum_{j=k}^{n} \frac{(-1)^{j-k} n!}{k!(j-k)!(n-j)!} \left(\frac{n-j}{n}\right)^r,$$

(b)

$$\lim_{n\to\infty} \left(\left(1-\frac{x}{n}\right)^n e^x\right)^{\log n} = 1, \quad \forall x \geq 0,$$

(c)

$$1 - x \leq e^{-x}, \text{ for } 0 \leq x \leq 1,$$

(d) $p_k \to e^{-\lambda}\frac{\lambda^k}{k!}$ provided r, n increases to infinity so that $ne^{-r/n} \to \lambda$.

[8+4+4+8]

(2) Let $X_j \sim \Gamma(\alpha_j, \lambda)$ be independent random variables for $j = 1, \cdots, k$. Let $T = \sum_{j=1}^{k} X_j$ and for $j = 1, \cdots, k-1$ let $Y_j = \frac{X_j}{T}$. Obtain the joint pdf of $Y = (Y_1, \cdots, Y_{k-1})$. Also compute the moment $\mathbb{E}Y_1^{\beta_1} \cdots Y_{k-1}^{\beta_{k-1}}$. **[10+2]**

(3) Let $X_1, X_2, \cdots$ be i.i.d random variables that takes three values $0, 1, 2$ with probabilities

$$P(X_1 = 0) = P(X_1 = 1) = P(X_1 = 2) = 1/3.$$

Let $X = \sum_{i=1}^{\infty} \frac{X_i}{3^i}$. Compute the characteristic function of X and conclude that X is uniformly distributed. **[12]**

(4) Let $X_1, X_2, \cdots$ be i.i.d nonnegative random variables such that $\mathbb{E}(X_1) = \infty$. Show that $\frac{\sum_{i=1}^{n} X_i}{n} \to \infty$ with probability one. (Hint: Consider the random variables $Y_{k,m} = X_k \wedge m$.)

[12]

(5) Let $h : S \to \mathbb{R}$ be a function. We say that h is harmonic at x if

$$h(x) = \sum_{y \in S} P(x, y)h(y).$$

Let P be the transition matrix of an irreducible Markov chain with a finite state space S. Let $B \subseteq S$ be a non-empty subset of the state space, and assume $h : S \to \mathbb{R}$ is a function harmonic at all states $x \notin B$. Prove or disprove

(a) If h is non-constant and $h(y) = \max_{x \in S} h(x)$, then $y \in B$.

(b) There exists $y \in B$ such that $h(y) = \max_{x \in S} h(x)$.

[8+8]

(6) Consider the Markov chain on $\{0, 1, \cdots, 5\}$ having transition matrix

	0	1	2	3	4	5
0	1/2	1/2	0	0	0	0
1	1/3	2/3	0	0	0	0
2	0	0	1/8	0	7/8	0
3	1/4	1/4	0	0	1/4	1/4
4	0	0	3/4	0	1/4	0
5	0	1/5	0	1/5	1/5	2/5

(a) Determine states which are transient and which are recurrent.

(b) Find $\rho_{\{0,1\}}(x), x = 0, \cdots, 5$. (Recall given a closed subset C of the state space $\rho_C(x) = P(T_C < \infty | X_0 = x)$ is the probability that a Markov chain starting at x eventually hits C.)

(c) Find the stationary distribution concentrated on each of the irreducible closed sets.

(d) Find $\lim_{n\to\infty} \frac{G_n(x,y)}{n}$, for $(x, y) \in \{(0, 1), (0, 2), (0, 3)\}$ ($G_n(x, y)$ is starting from x the expected number of visits by the chain to the state y upto time n.)

[5+5+5+5]

(7) Cleanliness **[4]**