INDIAN STATISTICAL INSTITUTE
CHENNAI CENTRE
M. Stat. (NB Stream) – Semester I
2015 – 2017
Linear Algebra and Linear Models
Final Examination (Linear Algebra)

Total Marks: 50 Maximum Marks: 45

Date: 11 September 2015 **Duration:** 3 hours

1. Let A, B and C be matrices of order $n \times n$, $n \times k$ and $k \times n$ respectively. Show that $|A+BC| = |A| \cdot |I_k + CA^{-1}B|$ if A is non-singular. [5]

2. (a) Show that $\langle A, B\rangle = \text{trace}(B^*A)$ is an inner product on $\mathbb{C}^{m\times n}$. [4]

   (b) When is $\|\mathbf{x}\| = \sum_{i=1}^{n} \alpha_i |x_i|$ a norm on $\mathbb{R}^n$, where $\mathbf{x} = (x_1, \ldots, x_n)^T$. [3]

   (c) Let f be any map from $\mathbb{R}^n$ to itself such that

   $$\|f(\mathbf{x}) - f(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\| \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n,$$

   where $\|\cdot\|$ is the norm induced by the canonical inner product on $\mathbb{R}^n$. Show that there exists an orthogonal matrix A of order n and a vector $\mathbf{c} \in \mathbb{R}^n$ such that $f(\mathbf{x}) = A\mathbf{x} + \mathbf{c}$ for all $\mathbf{x} \in \mathbb{R}^n$. [6]

3. (a) Let A and B be matrices of order $m \times n$ and $n \times m$ respectively. Then show that every non-zero eigenvalue α of AB is also an eigenvalue of BA with the same algebraic multiplicity. Also show that the geometric multiplicity of α with respect to AB and BA are equal. [4+3]

   (b) Prove that the minimal polynomial of $diag(A, B)$ is the L.C.M. of the minimal polynomials of A and B. [3]

   (c) State and prove the spectral theorem for real symmetric matrices. [1+4]

   (d) Let A be a square matrix over $\mathbb{C}$. Prove or disprove the following. (i) The number of singular values of A is equal to the rank of A. (ii) The number of non-zero characteristic roots of A is equal to the rank of A. [2+2]

4. (a) Let A be a symmetric matrix and P be a matrix with full column rank over $\mathbb{R}$. Show that A and PAP^T have the same rank and the same signature. [3]

   (b) Let A be a real symmetric matrix of order $n \times n$. Show that the set $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T A\mathbf{x} \leq 1\}$ is bounded iff A is positive definite. [5]

   (c) Show that a matrix A is positive definite iff $A = B^T B$ for some real non-singular matrix B. [5]