INDIAN STATISTICAL INSTITUTE
CHENNAI CENTRE
M.STAT First Year
2015-16 Semester I

Analysis I
Final Examination

Total Marks 60. Duration: 3 hours

Answer any 10 questions. All questions carry equal marks.

1. Let $\{f_n\}$ be a sequence of non-decreasing functions defined from $[0,1]$ to $[0,1]$. Suppose that $f_n(x)$ converges to $f(x)$ pointwise on $[0,1]$, where f is a continuous function. Does f_n converge uniformly to f on $[0,1]$? Justify your answer.

2. Let f and g be increasing real valued functions defined over $[a,b] \subset \mathbb{R}$, and g be continuous on $[a,b]$. Show that there exists $c \in [a,b]$ such that
$$\int_a^b f dg = f(a) \int_a^c dg + f(b) \int_c^b dg.$$

3. Let f be a continuous non-negative function defined over $[a,b]$, and
$$\int_a^b f(x)dx = 0.$$
Prove that $f(x) = 0$ for all $x \in [a,b]$.

4. Sketch the curve of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x^3 + 3x^2 + 1$.

5. Let $f : [0,1] \to \mathbb{R}$ be a continuous function, with $f(0) = f(1) = 0$. Suppose that f'' exists on $(0,1)$, with $f'' + 2f' + f \geq 0$. Show that $f(x) \leq 0$, for all $x \in [0,1]$.

6. Let X be a metric space and $f : X \to \mathbb{R}$ be a continuous function. Let $A = \{x \in X : f(x) = 4\}$. Is A closed / open / both / neither? Justify your answers.

7. Let $f : \mathbb{R} \to \mathbb{R}$ such that $|f(x) - f(y)| \leq \lambda |x - y|$ for all x, y, and for some real $\lambda < 1$. Let $\{a_n\}$ be a sequence of real numbers as follows: $a_1 \in \mathbb{R}$, and $a_{n+1} = f(a_n)$. Show that this is a convergent sequence and its limit is a fixed point of f.

8. Let X be a countable set. For each $n \geq 1$, let A_n be the set of all functions from X to $\{1, 2, \ldots, n\}$. Is $\bigcup_{n \geq 1} A_n$ countable? Justify your answer.

9. Does $\sum\limits_{n=1}^{\infty} (1/n)\cos(4n)$ converge? Justify your answer.

10. Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

    (a) If $B_n = \cup_{i=1}^n A_i$, then show that $\overline{B_n} = \cup_{i=1}^n \overline{A_i}$, $n = 1, 2, 3, \ldots$.

    (b) If $B = \cup_{i=1}^\infty A_i$, then show that $\overline{B} \supseteq \cup_{i=1}^\infty \overline{A_i}$. Can this inclusion be proper? Justify your answer.

11. Prove or disprove the following statement: Every closed subset of $\mathbb{R}$ is the intersection of a countable collection of open sets.

12. Let (X, d) be a metric space. If A is a connected subset of X and $A \subset B \subset \overline{A}$, then show that B is also connected.