Probability Theory
Mid Semestral Examination
Duration: 180 minutes
Maximum Score 90

(1) A social scientist wanted to know the percentage of drug users in a population he was studying. Realizing that people are not keen on answering direct questions on the subject, he devised the following procedure:

A big bag was prepared with many question slips. On 70% of the slips the question "Do you use drugs?" was written. On the remaining 30 % the question: "Is the last digit of your social security number even?" was written. Each subject randomly drew a slip from the bag, read it, responded to the interviewer with a "Yes" or "No?', and destroyed the slip. This way, the interviewer did not know which question the subject answered.

Assume that exactly fifty percent of the population has social security number ending with an even last digit, and that all subjects responded truthfully.

(a) It turned out that 44% of the subjects answered "yes". Give an estimate of the proportion of drug users in this population.

(b) What percentage of "yes" answers would have been obtained, had all the subjects been using drugs?

[8+4]

(2) Let $\{X_n\}$ be a sequence of nonnegative random variables increasing to X. For each n let $\{X_{nm}\}_m$ be a sequence of nonnegative random variables increasing to X_n. Let $Z_m = \max\{X_{nm} : 1 \le n \le m\}$. Show that $\mathbb{E}(Z_m) \uparrow \mathbb{E}(X)$. **[10]**

(3) Let X be a continuous nonnegative random variable such that

$$P(X > a)P(X > b) = P(X > a + b), \quad \forall a, b \in \mathbb{R}.$$

Show that X follows exponential distribution. **[12]**

(4) Let X_i be i.i.d Bernoulli(p) for $i = 1, \cdots$. Let $X = \min\{n : \sum_{i=1}^{n} X_i = r\}$. Show that

$$\mathbb{E}(e^{tX}) = \frac{(pe^t)^r}{[1-(1-p)e^t]^r}, \text{ for } t < -\log(1-p).$$

[15]

(5) Let $X_1, X_2, \cdots$ be i.i.d random variables that takes three values $0, 1, 2$ with probabilities

$$P(X_1 = 0) = p, P(X_1 = 1) = q, P(X_1 = 2) = r.$$

Let $X = \sum_{i=1}^{\infty} \frac{X_i}{3^i}$. Compute $\mathbb{E}(X)$ and $\mathbb{V}ar(X)$. Conclude that X is not uniformly distributed unless

$$p = q = r = 1/3.$$

[12]

(6) Let Ω be a set with n elements.

(a) Let $\mathcal{A}$ be the collection of all Boolean algebras of subsets of Ω.

(b) A partition of Ω is a collection of nonempty disjoint subsets $A_1, \cdots, A_k$ such that $\Omega = \cup_{j=1}^{k} A_j$. Let $\mathcal{P}$ be the collection of all possible partitions of Ω.

(c) Given a map $X : \Omega \to \mathbb{R}$, let $\sigma(X)$ be the smallest σ-algebra such that X is measurable.

(i) Show that there is a one to one, onto map $\Phi : \mathcal{A} \to \mathcal{P}$.

(ii) Let $X(\Omega) = \{x_1, \cdots, x_k\}$. Describe $\Phi(\sigma(X))$ in terms of the sets $X^{-1}(\{x_j\})$, $1 \leq j \leq k$ and prove your assertion .

(iii) Show that $Y : \Omega \to \mathbb{R}$ is $\sigma(X)$ measurable iff

$$\forall u, v \in \Omega, X(u) = X(v) \Longrightarrow Y(u) = Y(v).$$

[6+6+8]

Or

(i) Let $I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$. Compute I.I in polar coordinates and conclude that $I = \sqrt{2\pi}$. **[10]**

(ii) Let X, Y be random variables with finite second moment. Show that

$$|\mathbb{E}(X.Y)| \leq \sqrt{\mathbb{E}(X^2)\mathbb{E}(Y^2)}.$$

[10]

(7) Let $\{X_n\}$ be a sequence of random variables with $X_n \sim F_n$, for $n \geq 1$. Let X be a continuous random variable with distribution function F. Let $X_n \xrightarrow{d} X$. Show that $F_n(x_n) \to F(x)$ provided x_n converges to x and x is a continuity point of F.

[15]

(8) Cleanliness **[4]**