INDIAN STATISTICAL INSTITUTE

CHENNAI CENTRE

MidTerm Examination

Optimization Techniques

Answer any Five Questions

Total Marks : 30 Duration : 2.30 hours

(1) Let S_1 and S_2 be convex sets in R^n. Show that

$$\text{Conv}(S_1 \cap S_2) \subseteq \text{Conv}(S_1) \cap \text{Conv}(S_2).$$

Is Conv($S_1 \cap S_2$)=Conv(S_1)$\cap$ Conv(S_2) true in general?. If not, give a counter example.

(2) Identify the extreme points and extreme directions of the set S

$$S = \{X : -x_1 + 2x_2 \leq 4,\ x_1 - 3x_2 \leq 3,\ x_1, x_2 \geq 0\}$$

Represent the point (4,1) as a Convex Combination of extreme points and nonnegative combinations of extreme directions of S.

(3) Solve the following problem by the Simplex Method

$$\begin{aligned} \text{Maximise} \quad & 2X_1 + 3X_2 + 5X_3. \\ \text{Subject to} \quad & X_1 + 4X_2 - 2X_3 \leq 10 \\ & -X_1 + 2X_2 + 5X_3 \leq 3. \\ & X_1, X_2, X_3 \geq 0. \end{aligned}$$

What is the dual to this problem.

(4) Let A be a $p \times n$ matrix and B be a $(q \times n)$ matrix. Show that exactly one of the following systems has a solution.

$$\text{System 1} \quad AX < 0,\ BX = 0 \text{ for some} X \in R^n.$$

$$\text{System 2} \quad A^t u + B^t v = 0 \text{ for some non zero(u,v)}, u \neq 0, u \geq 0.$$

[Hint: You may use the following pair of primal and dual problem below and primal-dual feasibility relations

$$\text{P: Max } e^t u.$$

$$\text{Subject to} \quad A^t u + B^t v = 0,\ u \geq 0,\ v - \text{ unrestricted.}$$

$$\text{D: Min } 0^t x.$$

$$\text{Subject to} \quad Ax \geq e,\ Bx = 0,\ x - \text{ unrestricted.}]$$

(5) The Manager of a 24-hour super market has divided an average week day into 4 hour periods and figured out how many assistant he needs serving in each four hour period. His conclusions are given below

3.01-7.00 AM	7.01-11.0 AM	11.00AM-3.00 PM	3.01-7.00 PM	7.01-11.00PM	11.01PM-3.00PM
2	10	14	8	10	3

The manager's problem is to determine how the given number can be supplied for each period using the minimum number of assitants each day. If $x_1, x_2, \ldots, x_6$ are the number of assistants starting at 3 am, 7 am,..., 11 pm. respectively. Note that assistants reports for duty 3 am, 7 am, 11 a.m .. etc(every 4 hours) and their shift lasts 8 hrs.

(a) What is the objective function (in terms of decision variables $x_1, x_2, \ldots, x_6$)
(b) What are the inequality constraints?
(c) Write the dual of the problem.(in terms of decision variables $y_1, y_2, \ldots, y_6$)
(d) Write down complementary slackness condition for this problem.
(e) Verify that x=(0,14,0,8,2,2) is an optimal solution. (using complementary slackness and duality).

(6) Consider

$$\begin{aligned} \text{Maximise} \quad & x_2 - 3x_3 + 2x_4. \\ \text{Subject to} \quad & x_1 + x_2 + 3x_3 + 2x_4 + x_5 = 19 \\ & -6x_2 + 7x_3 + 8x_4 + x_5 + x_6 = 22 \\ & x_1 - x_2 + 2x_3 + 10x_4 + x_6 = 17 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \end{aligned}$$

Start with x_1, x_5, x_6 as your initial basis. Identify the initial basic feasible solution and perform simplex iterations.