On Multivariate Distribution and Quantile Functions, Ranks and Signs

a measure transportation approach

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Unlike the real line, the d-dimensional space $\mathbb{R}^d$, for $d \geq 2$, is not canonically ordered. As a consequence, such fundamental and strongly order-related univariate concepts as quantile and distribution functions, and their empirical counterparts, involving ranks and signs, do not canonically extend to the multivariate context. Palliating that lack of a canonical ordering has remained an open problem for more than half a century, and has generated an abundant literature, motivating, among others, the development of statistical depth and copula-based methods. We show here that, unlike the many definitions that have been proposed in the literature, the measure transportation-based ones introduced in Chernozhukov et al. (2017) enjoy all the properties (distribution-freeness and preservation of semiparametric efficiency) that make univariate quantiles and ranks successful tools for semiparametric statistical inference. We therefore propose a new *center-outward* definition of multivariate distribution and quantile functions, along with their empirical counterparts, for which we establish a Glivenko-Cantelli result. Our approach, based on results by McCann (1995), is geometric rather than analytical and, contrary to the Monge-Kantorovich one in Chernozhukov et al. (2017) (which assumes compactly supported distributions), does not require any moment assumptions. The resulting ranks and signs are shown to be strictly distribution-free, and maximal invariant under the action of transformations (namely, the gradients of convex functions, which thus are playing the role of order-preserving transformations) generating the family of absolutely continuous distributions; this, in view of a general result by Hallin and Werker (2003), implies preservation of semiparametric efficiency. The resulting quantiles are equivariant under the same transformations, which confirms the order-preserving nature of gradients of convex function.