Algebraic Structures

Homework 1

1. Let A, B, C and D be 4 non-empty sets. Prove or disprove: $A \subseteq B$ and $C \subseteq D$ imply $A \setminus C \subseteq B \setminus D$.

2. Let A be a non-empty set. Prove that the following are equivalent:

   (a) There exists an equivalence relation on A.

   (b) There exists a partition of the set A.

3. Give an example of an equivalence relation inducing an infinite number of partitions, each containing a finite number of elements.

4. Let A, B, C be 3 non-empty sets. Does there exist a bijection between the sets $C^{A \cup B}$ and $C^A \times C^B$? Justify your answer.

5. Show that the composition of two bijections is also a bijection.

6. Show that $\mathbb{Z} \times \mathbb{Z}$ is a countable set.

7. Let X be a set. Show that the following statements are equivalent:

   - For some n ≥ 0, there is a bijection $f : X \to [n]$, where $[n] = \{1, \ldots, n\}, n \geq 0$.
   - There is no proper subset Y of X such that there is a bijection $f : X \to Y$.

8. Show that countable sets are closed under countable union and finite cartesian product.

9. Let X be a non-empty set. Consider two sets A and B defined as follows:

   $$A = \mathcal{P}(X),\ B = \{f \mid f : X \longrightarrow \{0, 1\}\}.$$

   Does there exist a bijection between the sets A and B? Justify your answer.

10. Show that the set of all finite subsets of $\mathbb{N}$ is countable.

11. Show that the set of rational numbers is countable.

12. Prove that the set of all irrational numbers is uncountable.
13. Prove or disprove: The set $\{0, 1\} \times \mathbb{R}$ is uncountable.
14. Study the proof of Schröder-Bernstein theorem.