Algebraic Structures

Homework 2

1. Assume that the equation $xyz = e$ holds in a group G, where e denotes the identity element of G. Does it follow that $yzx = e$? Also $yxz = e$? Justify your answers.

2. Let S be any set. Prove that the law of composition defined by $ab = a$ is associative.

3. Determine the elements of the cyclic group generated by the matrix M, where

$$M = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

4. An nth root of unity is a complex number z such that $z^n = 1$. Prove that the nth roots of unity form a cyclic subgroup of $\mathbb{C}^\times$ of order n.

5. In the definition of subgroup, the identity element in H is required to be the identity of G. One might require only that H have an identity element, not that it is the same as the identity in G. Show that if H has an identity at all, then it is the identity in G. Show the analogous thing for inverses.

6. Let a, b be elements of an abelian group of orders m, n respectively. What can you say about the order of their product ab?

7. Prove that the additive group $\mathbb{R}^+$ of real numbers is isomorphic to the multiplicative group P of positive reals.

8. Let $f : G \to G'$ be an isomorphism of groups, let $g \in G$ and $f(g) = g'$. Prove that the orders of g and of g' are equal.

9. Prove that the set $Aut\ G$ of automorphisms of a group G forms a group, the law of composition being composition of functions.

10. Prove that the kernel and image of a homomorphism are subgroups.

11. Let $f : G \to G'$ be a surjective homorphism of groups and N be a normal subgroup of G. Prove that $f(N)$ is a normal subgroup of G' .