Algebraic Structures

Homework 3

1. Let S and T be two non-empty sets and $f : S \to T$. Prove that the nonempty fibres of a map$f$ form a partition of the domain S, where a fibre is a set of those elements of S which is mapped to a single element in T.

2. Let G be a group and $x, y \in G$. x is said to be a conjugate to y if there exists $g \in G$, such that $x = gyg^{-1}$.

   (a) Prove that the relation x conjugate to y in a group G is an equivalence relation on G.

   (b) Describe the elements a whose conjugacy class (= equivalence class) consists of the element a alone.

3. Prove that the distinct cosets in a group do not overlap.

4. Let H, K be subgroups of a group G of orders 3,5 respectively. Prove that $H \cap K = \{e\}$.

5. Show the following:

   (a) Every subgroup of index 2 is normal.

   (b) A subgroup of index 3 may not be normal.

6. Classify groups of order 6 by analyzing the following three cases.

   (a) G contains an element of order 6.

   (b) G contains an element of order 3 but none of order 6.

   (c) All elements of G have order 1 or 2.

7. Answer the following:

   (a) Prove that the square of an integer is congruent to 0 or 1 modulo 4.

   (b) What are the possible values of the square of an integer modulo 8?

8. Answer the following:

   (a) Prove that 2 has no multiplicative inverse modulo 6.

(b) Determine all integers n such that 2 has an inverse modulo n.

9. Let G be the group of invertible real upper triangular 2×2 matrices. Determine whether or not the following conditions describe normal subgroups H of G. If they do, use the First Isomorphism Theorem to identify the quotient group G/H.
(a) $a_{11} = 1$ (b) $a_{12} = 0$ (c) $a_{11} = a_{22}$ (d) $a_{11} = a_{22} = 1$.

10. Let P be a partition of a group G with the property that for any pair of elements A, B of the partition, the product set AB is contained entirely within another element C of the partition. Let N be the element of P which contains 1. Prove that N is a normal subgroup of G and that P is the set of its cosets.

11. Let $\mathbb{R}^{\times}$ denote the group of non-zero real numbers under multiplication. Identify the quotient group $\mathbb{R}^{\times}/P$, where P denotes the subgroup of positive real numbers.

12. Let $H = \{\pm 1, \pm i\}$ be the subgroup of $G = \mathbb{C}^{\times}$ of fourth roots of unity. Describe the cosets of H in G explicitly, and prove that G/H is isomorphic to G.