# Algebraic Structures

## Homework 4

1. Prove the following identities in an arbitrary ring $R$:

   (1) 0a=0. (2) -a = (-l)a. (3) (-a)b = -(ab).

2. In each case, decide whether the given structure forms a ring. If it is not a ring, determine which of the ring axioms hold and which fail:

   (a) $U$ is an arbitrary set, and $R$ is the set of subsets of $U$. Addition and multiplication of elements of $R$ are defined by the rules $A+B = A\cup B$ and $A.B = A \cap B$.

   (b) $U$ is an arbitrary set, and $R$ is the set of subsets of $U$. Addition and multiplication of elements of $R$ are defined by the rules $A+B = (A \cup B) \setminus (A \cap B)$ and $A.B = A \cap B$.

3. Describe the group of units in each ring:

   (a) $\mathbb{Z}/12\mathbb{Z}$. (b) $\mathbb{Z}/7\mathbb{Z}$. (c) $\mathbb{Z}/8\mathbb{Z}$. (d) $\mathbb{Z}/n\mathbb{Z}$.

4. Prove that the product set $R\times R$ of two rings is a ring with component-wise addition and multiplication:

   $$(a, a') + (b, b') = (a + b, a' + b') \text{ and } (a, a')(b, b') = (ab, a'b')$$

5. Prove that the units of the polynomial ring $\mathbb{R}[x]$ are the nonzero constant polynomials.

6. Determine the structure of the ring $\mathbb{Z}[x]/(x^2 + 3, p)$, where (a) $p = 3$, (b) $p = 5$.

7. Prove that an integral domain with finitely many elements is a field.

8. Let $R$ be an integral domain. Prove that the polynomial ring $R[x]$ is an integral domain.

9. Is there an integral domain containing exactly 10 elements?

10. A subset $S$ of an integral domain $R$ which is closed under multiplication and which does not contain 0 is called a multiplicative set. Given a multiplicative set $S$, we define $S$-fractions to be elements of the form $a/b$, where $b \in S$. Show that the equivalence classes of $S$-fractions form a ring.

11. Prove that in the ring $\mathbb{Z}[x]$, $(2) \cap (x) = (2x)$.

12. Let $R$, $R'$ be rings, and let $R \times R'$ be their product. Which of the following maps are ring homomorphisms? (a) $r \mapsto (r, 0)$ (b) $r \mapsto (r, r)$ (c) $(r_1, r_2) \mapsto r_1$ (d) $(r_1, r_2) \mapsto r_1 r_2$ (e) $(r_1, r_2) \mapsto r_1 + r_2$

13. (a) Is $\mathbb{Z}/10\mathbb{Z}$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$?

    (b) Is $\mathbb{Z}/8\mathbb{Z}$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$?

14. (a) Let $I, J$ be ideals of a ring $R$. Prove that $I \cap J$ is an ideal.

    (b) Show by example that the set of products $\{xy \mid x \in I, y \in J\}$ need not be an ideal.

15. (a) Prove that the set of finite sums of products of elements of $I$ and $J$ is an ideal. Denote the ideal by $IJ$.

    (b) Is$IJ \subseteq I \cap J$ ? Justify your answer.

    (c) Show by example that $IJ$ and $I \cap J$ need not be equal.