Algebraic Structures

Homework 5

1. Suppose we adjoin an element a to $\mathbb{R}$ (the real numbers) satisfying the relation $a^2 = 1$. Prove that the resulting ring is isomorphic to the product ring $\mathbb{R} \times \mathbb{R}$, and find the element of $\mathbb{R} \times \mathbb{R}$ which corresponds to a.

2. Let a be a unit in a ring R. Describe the ring $R' = R[x]/(ax - 1)$.

3. Let a be an element of a ring R, and let $R' = R[x]/(ax - 1)$ be the ring obtained by adjoining an inverse of a to R. Prove that the kernel of the natural homomorphism $f : R \to R'$ is the set of elements $b \in R$ such that $a^n b = 0$ for some $n > 0$.

4. Let $\mathbb{Q}$ be the field of rational numbers. Let $F = \mathbb{Q}(\{\sqrt{p} : p \in \mathbb{Z}, p \text{ is a prime}\})$. Show that F is an algebraic extension of $\mathbb{Q}$ and the dimension of this extension is infinite.

5. Let $\mathbb{Q}$ be the field of rational numbers, and $\mathbb{R}$ be the field of real numbers. Find an element $r \in \mathbb{R}$ such that $\mathbb{Q}(\sqrt{2}, \sqrt[3]{7}) = \mathbb{Q}(r)$.

6. Let $\mathbb{Q}$ be the field of rational numbers and $\mathbb{R}$ be the field of real numbers. Let $a, b \in \mathbb{R}$ be algebraic over $\mathbb{Q}$ of degree m and n, respectively. Suppose m and n are relatively prime. Show that $[\mathbb{Q}(a, b) : \mathbb{Q}] = mn$. Show that the result need not hold if m and n are not prime to each other.

7. Find a splitting field S of the polynomial $x^4 - 10x^2 + 21$ over $\mathbb{Q}$. Find $[S : \mathbb{Q}]$ and a basis of S over $\mathbb{Q}$.

8. Find a splitting field S of the polynomial $x^p - 1$ over $\mathbb{Q}$, where p is a prime number. Find $[S : \mathbb{Q}]$.

9. Find a splitting field of each of the polynomials over $\mathbb{Q}$.

   (a) $x^4 + 1$

   (b) $x^6 + x^3 + 1$

10. Prove that $x^3 + x + \overline{1}$ is irreducible in $(\mathbb{Z}/2\mathbb{Z})[x]$. Write down the addition and multiplication table for the field:

$$(\mathbb{Z}/2\mathbb{Z})[x]/(x^3 + x + \overline{1}).$$

Find a splitting field S for $x^3 + x + \overline{1}$ over $\mathbb{Z}/2\mathbb{Z}$. Find $[S : \mathbb{Z}/2\mathbb{Z}]$ and a basis for the field extension $S \supset \mathbb{Z}/2\mathbb{Z}$.

11. Prove that $x^3+x^2+\overline{1}$ is irreducible in $(\mathbb{Z}/2\mathbb{Z})[x]$. Write down the addition and multiplication table for the field:

$$(\mathbb{Z}/2\mathbb{Z})[x]/(x^3 + x^2 + \overline{1}).$$

Find a splitting field S for $x^3 + x^2 + \overline{1}$ over $\mathbb{Z}/2\mathbb{Z}$. Find $[S : \mathbb{Z}/2\mathbb{Z}]$ and a basis for the field extension $S/(\mathbb{Z}/2\mathbb{Z})$.

12. Are the splitting fields identified in Exercises 10 and 11 isomorphic as vector spaces over $\mathbb{Z}/2\mathbb{Z}$? Justify your answer.

13. Are the splitting fields identified in Exercises 10 and 11 isomorphic as fields? Justify your answer.

14. Factor the polynomial $x^8 - x$ over $\mathbb{Z}/2\mathbb{Z}$.