Elements of Algebraic Structures March 10, 2024

Lecture 10: Product of groups

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1 Topics for this lecture

In this lecture, we shall talk about the following

1. Product of groups
2. product of subgroups

2 Products of group

Let G and G' be two groups. The product set $G \times G'$, the set of pairs of elements (a, a') with a in G and a' in G' can be made into a group by the rule $(a, a') \cdot (b, b') = (ab, a'b')$. Now we will see whether $G \times G'$ forms a group or not.
The pair (e_G, e'_G) is the identity and the inverse of (a, a') is (a^{-1}, a'^{-1}). The associative law in $G \times G'$ follows from the fact that it holds in G and G'.
The group obtained in this way is called the product of G and G' and is denoted by $G \times G'$.

exercise Show that the following are homomorphisms

1. $f_1 : G_1 \to G_1 \times G_2$ defined by $g_1 \mapsto (g_1, g_2)$
2. $f_2 : G_2 \to G_1 \times G_2$ defined by $g_2 \mapsto (e_{G_1}, g_2)$
3. $f_3 : G_1 \times G_2 \to G_1$ defined by $(g_1, g_2) \mapsto g_1$
4. $f_3 : G_1 \times G_2 \to G_2$ defined by $(g_1, g_2) \mapsto g_2$.

Proposition 2.1 *Let H and K be subgroups of a group G and let $f : H \times K \to G$ be the multiplication map defined by $f(h, k) = hk$. Its image is the set $HK = \{hk : h \in H, k \in K\}$.*

(a) f is injective if and only if $H \cap K = \{1\}$.

(b) f is homomorphism from the product group $H \times K$ to G if and only if elements of K commute with elements of $H : hk = kh$.

(c) If H is a normal subgroup of G, then HK is a subgroup of G.

(d) f is an isomorphism from the product group $H \times K$ to G if and only if $H \cap K = \{1\}, HK = G$ and also H and K are normal subgroups of G.

Proof.

(a) If $H \cap K$ contains an element $x \neq 1$, then x^{-1} is in H, and $f(x^{-1}, x) = 1 = f(1,1)$, so f is not injective. Suppose that $H \cap K = \{1\}$. Let (h_1, k_1) and (h_2, k_2) be elements of $H \times K$ such that $h_1 k_1 = h_2 k_2$. WE multiply both sides of this equation on the left by h_1^{-1} and on the right by k_2^{-1}, obtaining $k_1 k_2^{-1} = h_1^{-1} h_2$. The left side is an element of K and the right side is an element of H. Since $H \cap K = \{1\},\ k_1 k_2^{-1} = h_1^{-1} h_2 = 1$. Then $k_1 = k_2,\ h_1 = h_2$, and $(h_1, k_1) = (h_2, k_2)$.

(b) Let (h_1, k_1) and (h_2, k_2) be elements of the product groups $H \times K$. The product of these elements in the product group $H \times K$ is $(h_1 h_2, k_1 k_2)$ and $f(h_1 h_2, k_1 k_2) = h_1 h_2 k_1 k_2$, while $f(h_1, k_1) f(h_2, k_2) = h_1 k_1 h_2 k_2$. These elements are equal if and only if $h_2 k_1 = k_1 h_2$.

(c) Suppose that H is normal subgroup. We note that KH is a union of left cosets kH with $k \in K$ and that HK is a union of right cosets Hk. Since H is normal, $kH = Hk$ and therefore $HK = KH$. Closure of HK under multiplication follows, because $HKHK = HHKK = HK$. Also $(hk)^{-1} = k^{-1} h^{-1}$ is in $KH = HK$. This proves closure of HK under inverses.

(d) Suppose that H and K satisfy the conditions given. Then f is both injective and surjective, so it is bijective. According to (b), it is an isomorphism if and only if $hk = kh$ for all h in H and k in K, and since H is normal, the right side is in H. Since $H \cap K = \{1\}, hkh^{-1}k^{-1} = 1$ and $hk = kh$. Conversely, if f is an isomorphism, one may verify the conditions listed in the isomorphic group $H \times K$ instead of in G.

□

Example (*) *$G = \mathbb{R}^\times$ is isomorphic to the product group $H \times K$, where $H = \{1, -1\}$ and $K =$\{positive real numbers\}.*
G is abelian, hence H and K both are normal subgroups of G, $HK = G$ and $H \cap K = \{1\}$. Therefore, Proposition 2.1(d) shows that G is isomorphic to the product group $H \times K$.

Proposition 2.3 (*) *There are two isomorphism classes of groups of order 4, the class of the cyclic group C_4 of order 4 and the class of the Klein Four Group, which is isomorphic to the product $C_2 \times C_2$ of two groups of order 2.*

Proof. Let G be a group of order 4. THe order of any element x of G divides 4, so there are two cases to consider:
*Case*1: G containes an element of order 4. Then G isa cyclic group of order 4.
*Case*2: Every element of G except the identity has order 2.
In this case, $x = x^{-1}$ for every element x of G. Let x and y be two elements of G. Then

xy has order 2. so $xyx^{-1}y^{-1} = (xy)(xy) = 1$. This shows that x and y commute, and since these are arbitrary elements, G is an abelian group. So every subgroup is normal. We choose distinct elements x and y in G and we let H and K be the cyclic groups of order 2 that they generate. Proposition 2.1(d) shows that G is isomorphic to the product group $H \times K$. □

Exercise* Let x be an element of order r of a group G, and let y be an element of G' of order s. What is the order of (x, y) in the product group $G \times G'$?