**Abstract Algebra** March 19 2024

# Lecture 15: On Integral Domain

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**Topics for this lecture**

In this lecture, we shall talk about the following

1. Proof of the theorem
2. Quotient Field

Let us begin this lecture with the proof of the following theorem.

**Theorem 1** *Any integral domain can be extended to a field.*

*Proof.* Let R be an integral domain. Now, consider s = R × R$\backslash\{0_R\}$. Now we define a relation ∼ on S as follows:-

$$(a, b) \sim (c, d) \text{iff ad = cd}$$

. Since it is trivial that ∼ is reflexive and symmetric. To show that ∼ is transitive, let us assume that (a, b) ∼ (c, d) and (c, d) ∼ (e, f). Hence, we can say that ad = bc and ce = df. Then

$$\begin{aligned} adcf &= bcde \\ \implies adcf - bcde &= 0 \\ \implies cd(af - be) &= 0 \end{aligned}$$

Then, either cd = 0 or (af - be) = 0. Now, analyze the following cases:-

- if cd = 0, then c = 0 as d ≠ 0. Therefore, a = c = e = 0 and so (a, b) ∼ (e, f).
- If cd ≠ 0 then af - de = 0. Therefore af = de implies that (a, b) ∼ (e, f).

Hence, ∼ is an equivalent relation. Then S can be partitioned into equivalent classes. Let a|b denote the equivalence class containing (a, b) i.e., $^a/_b$ = [(a, b)]. Let F = $^S/\sim$ ={$^a/_b$ | (a, b) ∈ R × R$\backslash\{0\}$}. Now, let us define + and · on F as follows:-

- $^a/_b + {}^c/_d = {}^{ad+bc}/_{bd}$
- $^a/_b + {}^c/_d = {}^{ac}/_{bd}$

□

We first need to check that + and . are well defined:-

- If $^{a_1}/_{b_1} = {}^{a_2}/_{b_2}$ and $^{c_1}/_{d_1} = {}^{c_2}/_{d_2}$, we need to show that $^{a_1}/_{b_1} + {}^{c_1}/_{d_1} = {}^{a_2}/_{b_2} + {}^{c_2}/_{d_2}$ and $^{a_1}/_{b_1}$ . $^{c_1}/_{d_1} = {}^{a_2}/_{b_2}.{}^{a_2}/_{b_2}$

**Excercise**

1. Prove the well-definedness + and .

Now, we have to check that (F, +, .) forms a field.

- Additive identity : $^0/_r$, where r $\in R \setminus \{0\}$.
- Multiplication identity : $^r/_r$, r $\in R \setminus \{0\}$
- Additive inverse of $^r/_s$ : -$^r/_s$, r $\in$ R, s $\in$ R$\setminus\{0\}$
- Multiplication inverse of $^r/_s$: $^s/_r$, r, s $\in$ R$\setminus\{0\}$.

**Excercise**

1. Prove that (F, +, .) forms a field.

So, we have constructed a field F using the integral domain R. Now, we need to show that R sits inside F, that is, we have to find an injective homomorphism $h : R \to F$.

Define $h : R \to F, r \mapsto {}^r/_1$.

- **h is injective:-** Take any a, b $\in$ R. Now, h(a) = h(b) implies that $^a/_1 = {}^b/_1$ implies that a. 1 = b.1 that is a = b.
- **h is homomorphism:-** Take any a, b $\in$ R. Now,

$$\begin{aligned} h(a+b) &= {}^{a+b}/_1 & \qquad h(a.b) &= {}^{ab}/_1 \\ &= {}^a/_1 + {}^b/_1 & &= {}^a/_1 . {}^b/_1 \\ &= h(a) + h(b) & &= h(a).h(b) \end{aligned}$$

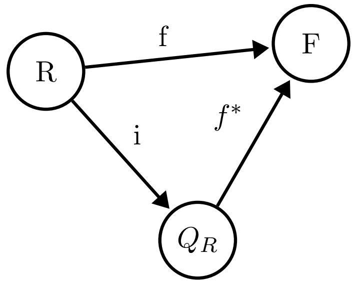
Thus, h is an injective homomorphism. This completes the proof. F is called a quotient field of R.

# 1 Quotient Ring:-

Let R be an integral domain. Then the quotient field $Q_R$, say, is the smallest field containing R in the sense that if F is any field such that there is an injective homomorphism $f : R \to F$. then there is an injective homomorphism.

$f^* : Q_R \to F$. How do we get this $f^*$?

Consider, the given diagram below:-




Given f, and the map h given above, which we denote here by i, we need to find $f^* : Q_R \to F$ by:

$$f^*({}^a\!/\!_b = f(a)[f(b)]^{-1}$$

- Since, b $\neq$ 0, f(b) $\neq$ 0, as f is injective. So, $[f(b)]^{-1}$ exists.
- Let us first check whether $f^*$ is well defined:

  Suppose ${}^a\!/\!_b = {}^c\!/\!_d$. To show that $f^*({}^a\!/\!_b) = f^*({}^c\!/\!_d)$. Now, we have ad = bc. So,

$$\begin{aligned} f(ad) &= f(bc) \\ \implies f(a)f(b) &= f(b)f(c) \\ \implies f(a)[f(b)]^{-1} &= f(c)[f(d)]^{-1} \end{aligned}$$

  Thus, $f^*$ is well defined.

**Excercise**

1. Prove that $f^*$ is an injective homomorphism.

Now, we need to show that

$$f^* \circ i = f$$

. To show that take any r $\in$R, therefore,

$$\begin{aligned} (f^* \circ i)(r) &= f^*(i(r)) \\ &= f^*({}^r\!/\!_1) \\ &= f(r)(f(1))^{-1} \\ &= f(r) \qquad \text{Since, f(1)} = 1_F \end{aligned}$$

Thus we have that $Q_R$, the quotient field of R is the smallest containing R.

**Excercise**

1. Let R be a commutative ring with identity and F be a field. Let $f : R \to F$ be an injective homomorphism. Then prove or disprove the following statement:-

$$f(1_R) = 1_F$$