Elements Of Algebraic Structure 07.03.2024

Lecture 15: More on Quotient Rings

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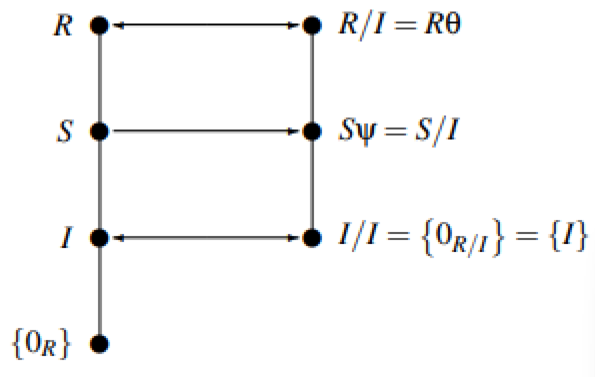
1 Correspondence Theorem for Rings

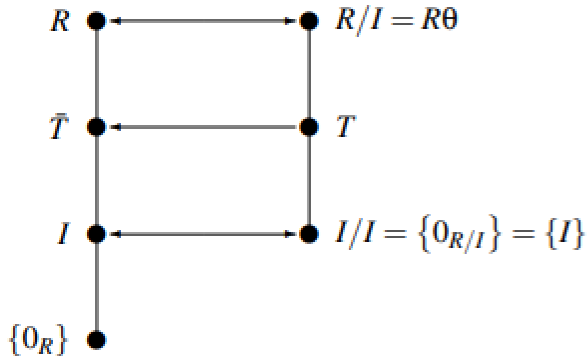
Correspondence Theorem for Rings Let I be an ideal of a ring R. There is a bijection Ψ from the set of subrings of R containing I and the set of subrings of R/I. The bijection preserves inclusion ($S_1 \subseteq S_2 \Rightarrow S_1\Psi \subseteq S_2\Psi$), and ideals of R containing I correspond to ideals of R/I.

Proof Define $S\Psi = \{I + s : s \in S\} = S/I$, for subrings S of R containing I. Let $\tilde{\theta}$ be the canonical homomorphism for I. Given S, let $\tilde{\theta}$ be the restriction of θ to S. Then $\tilde{\theta}$ is a homomorphism, and $Im(\tilde{\theta}) = s\theta : s \in S = I + s : s \in S = S\Psi$. But $Im(\tilde{\theta})$ is a subring of R/I, so $S\Psi$ is a subring of R/I.

Given $S\Psi$, we can recover S as the union of the cosets of I which are elements of $S\Psi$, so Ψ is one-to-one. Clearly, Ψ preserves inclusion.



Let T be a subring of R/I. Put $\tilde{T} = \{r \in R : I + r \in T\}$. We know from the group theory that $(\tilde{T}, +)$ is a subring of $(R, +)$ containing I. If r_1, r_2 are in $\tilde{T}$ then $I + r_1 \in T$ and $I + r_2 \in T$, so $(I + r_1)(I + r_2) \in T$, so $I + r_1r_2 \in T$, so $r_1r_2 \in \tilde{T}$. Hence $\tilde{T}$ is a subring of R. Clearly, $\tilde{T}\Psi = T$. Therefore Ψ is onto.



$$\begin{aligned}
S\Psi \triangle R/I &\iff (I+s)(I+r) \in S\Psi \text{ for all s in S and all r in R} \\
&\text{and } (I+r)(I+s) \in S\Psi \text{ for all s in S and all r in R} \\
&\iff I + sr \in S\Psi \text{ and } I + rs \in S\Psi \text{ for all s in S and all r in R} \qquad (1) \\
&\iff sr \in S \text{ and } rs \in S \text{ for all s in S and all r in R} \\
&\iff S \triangle R.
\end{aligned}$$