Elements of Algebraic Structures	January 16 2024

Lecture 3: Cardinality of Sets

Lecture: Sujata Ghosh *Scribe: Basudeb Roy, Shramana Dey*

1 Topics for this lecture

In this lecture, we shall talk about the following

1. Cardinality of Sets
   (a) Finite Sets
   (b) Infinite Sets
   (c) Countable Sets
   (d) Uncountable Sets

2 A Few Definitions

Definition 2.1 (Natural Number) *The set of all positive numbers from 1 to infinity. In this course, we assume the set of natural numbers includes 0.*

$$\mathbb{N} = \{0, 1, 2, \ldots\}$$

Definition 2.2 (Integer Number) *Integers are a set of whole numbers that include both positive and negative numbers, as well as zero.*

$$\mathbb{Z} = \{x \mid x \in \mathbb{N}\} \cup \{-x \mid x \in \mathbb{N}\}$$

Positive Integers: *Integers greater than zero, such as 1, 2, 3, and so on and is represented by* $\mathbb{Z}^+$.
Negative Integers: *Integers lesser than zero, such as -1, -2, -3, and so on and is represented by* $\mathbb{Z}^-$.

Definition 2.3 (Rational Number) *The set of all numbers that can be represented in a fractional form or finite, repeating decimals.*

$$\mathbb{Q} = \left\{ \frac{a}{b} \,\middle|\, a \in \mathbb{Z} \text{ and } b \in \mathbb{N} \setminus \{0\} \right\}$$

For example: $\frac{3}{4}, \frac{22}{7}, 0.5, \cdots$
The set of ***positive rational numbers*** *is represented by* $\mathbb{Q}^+$ *and the set of* ***negative rational numbers*** *is represented by* $\mathbb{Q}^-$
Note: *If we construct the whole set of rational numbers as defined earlier we get a set with duplicate or redundant numbers, so we assume 'a' and 'b' to be co-prime.*

Definition 2.4 (Irrational Number) *The set of all numbers that cannot be represented in a fractional form or finite, repeating decimals. Their decimal expansions go on forever without repeating.*
For example: $2\sqrt{2}, \pi, 1.6180339887498948482045868 34\ldots, \cdots$

Definition 2.5 (Real Number) *The set of all numbers that are either rational or irrational.*

$$\mathbb{R} = \{x | x \in \textit{ rational number or } x \in \textit{irrational number}\}$$

3 Review

1. From the previous lecture, we gathered the knowledge that, an **equivalence relation** on the collection or set gives us a **partition of the collection**.
2. Two sets A and B are said to be **equipotent** if there is a bijection between A and B.
3. The question of how we can distinguish two infinite sets in terms of 'the number of elements' present in those sets, needs to be addressed. The number of elements in a set is known as the cardinality of a set. For a finite set $A = \{a, b, c, d, e\}$ we say cardinality of the set A, $|A| = 5$. Then what can be said about the number of elements in the infinite set? We will discuss it in this lecture.

4 Cardinality of Sets

So, we learnt that a larger collection or set can be partitioned into smaller sets based on equivalence relation.

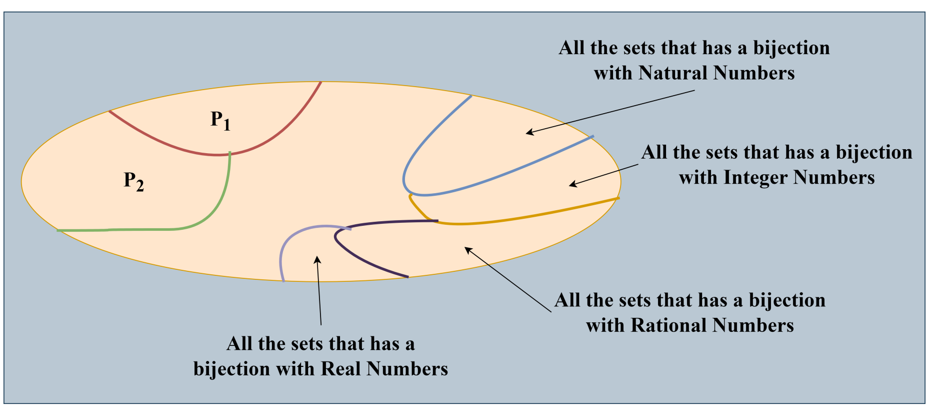

Figure 1: Partition in a larger Set

A set of numbers could be partitioned as shown in Figure 1.
Note: Technically the above the image is wrong as set of natural numbers and integers has a bijection between them, but let's keep this for a later discussion.
Now, the question arises:

1. Does $\mathbb{N}$ and $\mathbb{Z}$ belong to the same partition?
2. Does $\mathbb{Z}$ and $\mathbb{Q}$ belong to the same partition?
3. What is the pairwise relation between set $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$? Do they belong to the same partition? In other words, does there there exist a bijection?

When two sets belong to the same class, it implies they have the same number of elements. Consider set A and set B. Bijection means there is an injection and surjection relation between A and set B, thus implying the number of elements to be equal. In other words, exactly one element of A is related to one element of B.
For now, let's say two sets have the same cardinality if there exist a bijection between them.

Lemma 4.1 *Two sets have the same cardinality if there exists a bijection between them.*

Proof. If a bijection exists, then each element of one set can be mapped to a unique element of the other and this mapping exists both ways, hence they have the same cardinality. For every image, there exists a pre-image and vice-versa. □

Now, it is valid to ask, is there only one kind of infinite set?
This leads to our next section of countable and uncountable sets.

5 Countable and Uncountable Sets

Definition 5.1 (Countable Set) *As we will see later in this section, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ have a bijection relation, this can belong to the same equivalence class. This class is called the countable set.*
A set X is countable if there is a bijection between $\mathbb{N}$ and X.

Definition 5.2 (Uncountable Set) *A set X is said to be uncountable if it is not finite and not countable.*

5.1 Examples of Countable Set

1. $\mathbb{N}$ is countable:
$$f : \mathbb{N} \to \mathbb{N}$$
The above is an identity map.
2. Infinite subset of $\mathbb{N}$ is countable.
$$f : \text{subset of } \mathbb{N} \to \mathbb{N}$$
To show this, let's consider a more general statement.

Proposition 5.3 *Consider a countable set A. Let E be an infinite subset of A. Show that E is also countable.*

Proof. Given, A is a countable set and $E \subseteq_{inf} A$.
Let us enumerate the elements of A in a sequence $\{x_n\}_{n\in\mathbb{N}}$ of distinct elements.
$x_1, x_2, x_3, \cdots$
Construct a sequence $\{n_k\}_{k\in\mathbb{N}}$ as follows. Let n_1 be the smallest positive integer such that $x_{n1} \in E$. Let n_2 be the smallest positive integer such that $x_{n2} \in E$ and greater than n_1. Given, $n_1, n_2, n_3, \cdots, n_k$, let n_{k+1} be the smallest positive integer greater than n_k with $x_{nk+1} \in E$.
Now, define a map

$$f : \mathbb{N} \to E \text{ where } f(k) = x_{nk}$$

This gives us the required bijection. □

Note: In some sense, countable sets refer to the smallest infinity.

3. $\mathbb{Z}$ is countable.

Proposition 5.4 *Consider the set of integers, $\mathbb{Z}$. Show that $\mathbb{Z}$ is also countable.*

Proof. Let us consider the mapping of 0 to 1, −1 to 2, 1 to 3, −2 to 4, 2 to 5, and so on.
Let's define a map

$$f : \mathbb{Z} \to \mathbb{N} \setminus \{0\} \text{ where } f(-n) = 2n, f(n) = 2n + 1 \text{ and } f(0) = 1$$

This gives us the required bijection. Since a subset of a countable subset is also countable, so $\mathbb{N} \setminus \{0\}$ is also countable. Thus $\mathbb{Z}$ and $\mathbb{N}$ belong to the same equivalence class and $\mathbb{Z}$ is countable. □

4. Union of two countable sets is countable.

Exercise Verify the above statement.

Proposition 5.5 *Show that the union of two countable sets is countable.*

Proof. To show that the union of two countable sets is countable, we need to establish a bijection between the union of the two sets and the set of natural numbers $\mathbb{N}$. Let's consider two countable sets A and B. Countability implies the existence of bijections from each set to N. Let

$$f : A \to \mathbb{N} \text{ and } g : B \to \mathbb{N}$$

where f and g are bijective.

Now define

$$h : A \cup B \to \mathbb{N}$$

1. $h(x) = 2f(x) + 1$, where $x \in A$, is a bijection to natural number because f is a bijection from A to $\mathbb{N}$.
2. $h(y) = 2g(y)$, where $y \in B$, is a bijection to natural number because f is a bijection from B to $\mathbb{N}$.
The function h maps each element of the union $A \cup B$ to a natural number based on whether the element is in set A or set B. The elements of set A are mapped to odd natural numbers, while elements of set B are mapped to even natural numbers. This function is well-defined because each element in the union is in either A or B, but not both (since we assume that the sets are disjoint). It is also injective (one-to-one) because f and g are injective on their respective sets, and it is surjective (onto) because f and g are surjective. Therefore, h is a bijection between $A \cup B$ and $\mathbb{N}$. This proves that the union of two countable sets is countable. □

5. Union of a finite set and a countable set is countable.

Exercise Verify the above statement.

Proposition 5.6 *Show that the union of a finite set and a countable set is countable.*

Proof. To show that the union of a finite set and a countable set is countable, we need to establish a bijection between the union of the two sets and the set of natural numbers $\mathbb{N}$. Let's consider two countable sets A and B. Countability implies the existence of bijections from each set to N. Finite implies it contains a finite number of elements. Let

$$f : A \to \mathbb{N} \text{ and } g : B \to \mathbb{N}$$

where f and g are bijective. And,

$$|B| = k; \text{ where } k \in \mathbb{N}$$

Now define

$$h : A \cup B \to \mathbb{N}$$

1. $h(x) = f(x) + k$, where $x \in A$, is a bijection to natural number because f is a bijection from A to $\mathbb{N}$.
2. By the well-ordering principle, since B is finite but nonempty, it contains the least element. Let's enumerate the elements of B by n_p where $p \leq k$. Let's represent the elements of B by a_n. Therefore the least element in B is a_{n1}. a_{n2} is the least element in B greater than a_{n1}. a_{n3} is the least element in B greater than a_{n2} and so on. Then, $h(a_{np}) = n_p$, where $a_{np} \in B$, a_{np} is the least element in B greater than a_{np-1} and $p \geq 1$.
The function h maps each element of the union $A \cup B$ to a natural number based on whether the element is in set A or set B. The elements of set A are mapped to itself

in the set of natural numbers with an offset k, while elements of set B are mapped to the first k natural numbers. This function is well-defined because each element in the union is in either A or B, but not both (since we assume that the sets are disjoint). It is also injective (one-to-one) because f and g are injective on their respective sets, and it is surjective (onto) because f and g are surjective. Therefore, h is a bijection between $A \cup B$ and $\mathbb{N}$. This proves that the union of a finite set and a countable set is countable. □

Exercise* Show that the intersection of two countable sets is countable.

6. $\mathbb{Q}$ is countable.

Proposition 5.7 *Consider the set of rational numbers, $\mathbb{Q}$. Show that $\mathbb{Q}$ is also countable.*

Proof. We will show that $\mathbb{Q}$ can be put in one-to-one correspondence with the set of positive integers, proving that $\mathbb{Q}$ is countable.

We write,

$$\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$$

Every rational number can be expressed as a fraction $\frac{p}{q}$, where p and q are integers with no common factors other than 1, and q is not equal to 0.

Whereas, the positive rational number can be represented as,

$$f : \mathbb{Q}^+ \to \mathbb{N} \times \mathbb{N}$$

$$\mathbb{Q}^+ = \left\{ \frac{a}{b} \,\middle|\, a \in \mathbb{N} \setminus \{0\} \text{ and } b \in \mathbb{N} \setminus \{0\} \right\}$$

Let's list all the positive rational numbers in a grid, where each row corresponds to a different positive denominator:

$$\begin{array}{ccccc} \frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \cdots \\ \frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \cdots \\ \frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Now, we traverse the grid diagonally, listing each fraction:

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \frac{1}{5}, \frac{2}{4}, \ldots$$

It is to be noted, that every rational number is included in this list. Thus, we have established a one-to-one correspondence between the set of positive rational numbers

$\mathbb{Q}^+$ and the set of positive integers that is the set of natural number $\mathbb{N}$, showing that $\mathbb{Q}^+$ is countable.

The rational numbers listed above are redundant or contain duplicate values. To eliminate the redundant elements, p and q must be co-prime, that is it has no common factor.

The negative rational number can be represented as,

$$f : \mathbb{Q}^- \to \mathbb{Z}^- \times \mathbb{N}$$

$$\mathbb{Q}^- = \left\{ -\frac{a}{b} \,\middle|\, a \in \mathbb{Z}^- \text{ and } b \in \mathbb{N} \setminus \{0\} \right\}$$

Let's list all the negative rational numbers in a grid, where each row corresponds to a different positive denominator:

$$\begin{matrix} -\frac{1}{1} & -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} & \cdots \\ -\frac{2}{1} & -\frac{2}{2} & -\frac{2}{3} & -\frac{2}{4} & \cdots \\ -\frac{3}{1} & -\frac{3}{2} & -\frac{3}{3} & -\frac{3}{4} & \cdots \\ -\frac{4}{1} & -\frac{4}{2} & -\frac{4}{3} & -\frac{4}{4} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

The above numbers can be traversed diagonally as shown before and thus we can similarly show, a one-to-one correspondence between the set of negative rational numbers $\mathbb{Q}^-$ and the set of positive integers that is the set of natural number $\mathbb{N}$, showing that $\mathbb{Q}^-$ is countable.

So, $\mathbb{Q}^+$ and $\mathbb{Q}^-$ are shown to be countable, and 0 is a finite set.

As shown in the previous section, the union of two countable sets is countable, so $\mathbb{Q}^+ \cup \mathbb{Q}^-$ is countable.

Again, the union of a finite and a countable set is countable, so $\mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$ is countable.

Thus $\mathbb{Q}$ is countable. This completes the proof. $\square$