Elements of Algebraic Structures January 19 2024

Lecture 4: More on Countable Sets

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1 Topics for this lecture

In this lecture, we shall talk about the following

1. Operations on the countable set.
2. Notations.
3. Definitions.
4. Schröder-Bernstein Theorem.

2 Operations on countable set

2.1 Union on countable sets

Theorem 2.1 *Union of finitely many countable sets is countable.*

Proof. **Base Case (Two Countable Sets):** Let A and B be two countable sets. This means there exist bijections $f : \mathbb{N} \to A$ and $g : \mathbb{N} \to B$.
Consider the union $A \cup B$. Define a function $h : \mathbb{N} \to A \cup B$ as follows:

$$h(x) = \begin{cases} f\left(\frac{x}{2}\right) & \text{if } x \text{ is even} \\ g\left(\frac{x+1}{2}\right) & \text{if } x \text{ is odd} \end{cases}$$

This function h is a bijection from $\mathbb{N}$ to $A \cup B$. Therefore, the union of two countable sets is countable.
Inductive Step (Finitely Many Countable Sets): Assume that the union of n countable sets is countable for some positive integer n. That is, if $A_1, A_2, \ldots, A_n$ are countable sets, then $A_1 \cup A_2 \cup \ldots \cup A_n$ is countable.
Now, consider $n + 1$ countable sets: $A_1, A_2, \ldots, A_n, A_{n+1}$. By the inductive assumption, the union $A_1 \cup A_2 \cup \ldots \cup A_n$ is countable.
Applying the base case result to this countable set and A_{n+1}, we can conclude that $(A_1 \cup A_2 \cup \ldots \cup A_n) \cup A_{n+1}$ is countable.
By mathematical induction, we have shown that the union of finitely many countable sets is countable.

□

Theorem 2.2 *Union of countably many countable sets is also countable.*

Proof. Let $\{A_n : n \in \mathbb{N}\}$ be a countable collection of countable sets. Let A $= \cup_{n\in\mathbb{N}} A_n$. To show that A is countable, let A_n's be pairwise disjoint.
Now, we have that each A_n can be written as: $\{a_{n1},\ a_{n2},\ a_{n3},\ \cdots\}$, that is
$A_n = \{a_{nk}\}_{\mathbb{R}\in\mathbb{N}}$. then we have the following arrangement for the members of A:

$$\begin{matrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{matrix}$$

Now, we traverse the grid diagonally, listing each element:

$$a_{11}, a_{21}, a_{12)} a_{31}, a_{22}, a_{13}, \cdots$$

. This enumeration gives a bijection between $\mathbb{N}$ and A. So A is countable. □

2.2 Product of countable sets

Theorem 2.3 *Let A and B be two countable sets, then $A \times B$ is countable.*

Proof. Let A $=\{a_1, a_2, \cdots\}$ and B $= \{b_1, b_2, \cdots\}$, then we can list all the rational numbers in a grid as follows:-

$$\begin{matrix} (a_1, b_1) & (a_1, b_2) & (a_1, b_3) & \cdots \\ (a_2, b_1) & (a_2, b_2) & (a_2, b_3) & \cdots \\ (a_3, b_1) & (a_3, b_2) & (a_3, b_3) & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{matrix}$$

Now, we traverse the grid diagonally, listing each element:

$$(a_1, b_1), (a_2, b_1), (a_1, b_2), (a_3, b_1), (a_2, b_2), (a_1, b_3), \ldots$$

this enumeration gives a bijection between **N** and $(A \times B)$. So, $(A \times B)$ is countable. □

Theorem 2.4 *Let $A_1, A_2, A_3, \cdots, A_n$ be n countable sets, where $n \geq 1$, then $A_1 \times A_2 \times A_3 \times \cdots \times A_n$ is also countable.*

Proof. The proof will be done by using induction on n.
Base Case:- when n = 2, using the theorem 2.3 we can prove that $A_1 \times A_2$ is also countable.
Inductive Hypothesis:- Let us assume that this theorem is true for any n-1 countable sets, i.e., $A_1, A_2, A_3, \cdots, A_{n-1}$ be n countable sets, where $n \geq 1$, then $A_1 \times A_2 \times A_3 \times \cdots \times A_{n-1}$ is also countable.
Induction Step:- Let $A_1 = \{a_{11}, a_{12}, \cdots\}$, $A_2 = \{a_{21}, a_{22}, \cdots\}$, $\cdots$, $A_n = \{a_{n1}, a_{n2}, \cdots\}$ then we can list all the rational numbers in a n-dimensional grid. By applying the diagonalization argument we can say that there is a bijection between $\mathbb{N}$ and $A_1 \times A_2 \times \cdots \times A_n$. So, $A_1 \times A_2 \times \cdots \times A_n$ is countable. □

Let $A_1, A_2, A_3, \cdots$ be a countable collection of countable sets. then the natural question comes into that what can we say about $\prod_{n \in \mathbf{N}} A_n$ i.e., the countable product of countable sets?

Let us take an example. Consider the set A = {0, 1}and also let $B = \prod_{n \in \mathbf{N}} A_n$ where $A_n = A$ for all n.

Lemma 2.5 *B is not a countable set.*

Proof. We will prove it by contradiction. Suppose B is countable. Then $B = \{b_1, b_2, b_3, \cdots\}$. Now, each b_i is a sequence of 0's and 1's then $b_i = (b_{i1}, b_{i2}, b_{i3}, \cdots)$, where $b_{ij} \in \{0, 1\}$ for all $j \in \mathbf{N}$.

Let c be a sequence of 0's and 1's defined as follows:-

$$c = \begin{cases} 0, & b_{ii} = 1 \\ 1, & b_{ii} = 0, i \in \mathbf{N} \end{cases}$$

Then, $c \neq b_i$ for any $i \in \mathbf{N}$. Thus, $c \notin B$ is a contradiction. Hence, our assumption that B is countable cannot be true. Hence, B is uncountable. □

Diagonalization argument

A set S is called **COUNTABLY INFINITE** if there is a bijection between S and ℕ. That is, you can label the elements of S 1, 2, . . . so that each positive integer is used exactly once as a label. In the year 1895, **Georg Cantor** proved this fact by showing that the set of real numbers is not countable, which is famously known as **"Diagonalization argument"**. In the lemma 2.5 we use this argument.

Time for a little exercise

Exercise Prove that $\mathbf{R}$ is uncountable.

Before proceeding let us fix some notations.

Notations:-

1. When we have $\prod_{n \in \mathbb{N}} A_n$, where $A_n = A$ for all n, we denote $\prod_{n \in \mathbb{N}} A_n$ by $A^{\mathbb{N}}$.
2. X^Y denotes the set of all functions from Y to X.
3. Any tuple over A, which is countable in size can be represented by a function f: $\mathbb{N} \to A$.
4. Above mention is also held for any indexing set I. If we consider the collection $\{A_i$: i ∈ I$\}$, $A_i = A$ for all $i \in I$, $\prod_{i \in I} A_i$ is also given by A^I.
   - Example:- A = {0, 1} and indexing set I = ℕ, then $\prod_{n \in \mathbb{N}} A_n$, $A_i = A$ for all $i \in \mathbb{I}$ is also given by $2^{\mathbb{N}}$ which is an uncountable.

Exercise Prove that there is a bijection between $2^{\mathbb{N}}$ and $\mathbb{R}$ is uncountable.

3 Definition:-

1. **Cardinality:-** Let A be any set. We denote cardinality of A by $|A|$. For example:-
   (a) $|\mathbb{N}|$ denote the cardinality of $\mathbb{N}$.
   (b) $\mathbb{R}$ denote the cardinality of $\mathbb{R}$.
   (c) Let $|\mathbb{N}| = \aleph_0$ and $|\mathbb{R}| = c$, then we have $2^{\aleph_0} = c$
2. If there is an injection from set A to set B, then we denote it by:$|A| \geq |B|$.
3. $<$ denotes the strict order.

Proposition 3.1 ***Schröder-Bernstein Theorem:*** *If there exist injective functions $f : A \to B$ and $g : B \to A$, then there exists a bijective function $h : A \to B$.*

Proof. Injection from A to B: $f : A \to B$ is injective, meaning that for any distinct elements $x_1, x_2 \in A$, $f(x_1) \neq f(x_2)$.
Injection from B to A: $g : B \to A$ is injective, meaning that for any distinct elements $y_1, y_2 \in B$, $g(y_1) \neq g(y_2)$.
Construction of Bijection: Consider the composition $g \circ f : A \to A$ and $f \circ g : B \to B$. Since these compositions are injective, they are also surjective because they have the same cardinality as the domain and codomain.
Bijections and the Inverse: Because $g \circ f : A \to A$ is surjective, it has an inverse $h : A \to B$ such that $h \circ (g \circ f) = \text{Id}_A$, where Id_A is the identity function on A. Similarly, because $f \circ g : B \to B$ is surjective, it has an inverse $k : B \to A$ such that $k \circ (f \circ g) = \text{Id}_B$.
Establishing the Bijection: Define $h : A \to B$ by $h = g \circ f$. Then h is a bijection because:

- *Injectivity:* h is injective because g and f are injective, and the composition of injective functions is injective.
- *Surjectivity:* h is surjective because g and f are surjective, and the composition of surjective functions is surjective.

This completes the proof. □