Elements of Algebraic Structures	March 10, 2024

Lecture 9: Cosets and Quotient groups

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1 Topics for this lecture

In this lecture, we shall talk about the following

1. Some examples of coset
2. First Isomorphism Theorem
3. More on Quotient groups

Proposition 1.1 *Proof of a finite group of prime order is a cyclic group*

Proof. Let $|G| = p$, a prime number. So, there are non-identity element in G. Take any such element g say. Consider the cyclic sub-group $\langle g \rangle$ generated by g.
Then, by Lagrange's theorem $|\langle g \rangle| \mid |G|$. so, $|\langle g \rangle| = 1 \text{or } p$. Since $g \neq e_g, |\langle g \rangle| = p$. But $\langle g \rangle \subseteq G$ and hence, $\langle g \rangle = G$.
Thus G is a cyclic group.

□

Problem 1 *Any group with prime p as it's order is cyclic. What about any group of order p^2?*

Solution: Group of order p^2 may not be cyclic. Consider the group of order 4:
$\left(\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \}, \cdot \right)$, where all non identity elements are of order 2, hence not cyclic. Klein's 4-group also not cyclic.

2 Some examples of cosets

Example *Consider the group $(\mathbb{Z}, +)$. We know that the subgroups are of the form $(n\mathbb{Z}, +)$, where $n \in \mathbb{Z}$. What are the cosets of $(5\mathbb{Z}, +)$?*
Cosets are $0 + 5\mathbb{Z},\ 1 + 5\mathbb{Z},\ 2 + 5\mathbb{Z},\ 3 + 5\mathbb{Z},\ 4 + 5\mathbb{Z}$.

Exercise Show that the above collection of sets partitions $\mathbb{Z}$.

Problem 2 *What are the cosets of $n\mathbb{Z}$ in $\mathbb{Z}$, $n \geq 1$?*

Solution: The collection of cosets of $n\mathbb{Z}$ in $\mathbb{Z}$ is $\{a + n\mathbb{Z} : 0 \leq a \leq n-1\}$.
Another way of writing this set is : $\{[0], [1], \ldots, [n-1]\}$
We will denote the set as $\mathbb{Z}_n$ or $\mathbb{Z}/n\mathbb{Z}$.

Exercise Does $\mathbb{Z}_n$ form a group under some operation?

Exercise Does the cosets of ker$f$ in the group G form a group under certain operation?

Exercise Take any group and take any subgroup H of G would the cosets of H in G always formal group under the operation we considered earlier? If not, what condition should we impose on subgroups to make this operation on cosets work?

3 Quotient Groups:

Let G be a group and H be a normal subgroup of G. The group formed the cosets of H in G, denoted by G/H, is called a quotient group of H in G. The group operation is given by: $aH * bH = abH$ for all $a, b \in G$.

Example

1. $(\mathbb{Z}/n\mathbb{Z}, +_n) : (a + n\mathbb{Z}) +_n (b + n\mathbb{Z}) = (a + b) + n\mathbb{Z}$.
2. $(G/kerf, *) : (a \cdot kerf) * (b \cdot kerf) = (a \cdot b)kerf$.

4 First Isomorphism Theorem

Theorem 4.1 *Let G and G' be two groups and $f : G \to G'$ be an epimorphism. Then, there is an isomorphism between $G/kerf$ and G'.*

Proof. Let $H = kerf$. Define $h : G/H \to G'$ by $aH \mapsto f(a)$.
We would like to show that h is an isomorphism:

$$\begin{aligned} h(aH * bH) &= h(abH) \\ &= f(ab) \\ &= f(a)\ f(b) \\ &= h(aH)\ h(bH) \end{aligned}$$

Thus, h is a homomorphism.
h is surjective:
Let $g' \in G'$. Then there is $g \in G$ such that $f(g) = g'$. Thus, $h(gH) = g'$. Hence, h is surjective.
h is injective:
Suppose $aH, bH \in G/H$ such that

$h(aH) = h(bH)$
$\implies f(a) = f(b)$
$\implies f(a)(f(b))^{-1} = e'_G$
$\implies f(ab^{-1}) = e'_G$
$\implies ab^{-1} \in H$
$\implies a \in bH$
$\implies aH = bH$
This completes the proof. □

Example *Show that $GL_n(\mathbb{R})/SL_n(\mathbb{R})$ is isomorphic to $\mathbb{R}^\times$.*
Consider the map $f : GL_n(\mathbb{R}) \to \mathbb{R}^\times$ defined by $f(A) = det\ A$, for all $A \in GL_n(\mathbb{R})$. We can easily verify that f is a surjective group homomorphism. The kernel of the map is $SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : det\ A = 1\}$. Hence by First Isomorphism Theorem, the quotient $GL_n(\mathbb{R})/SL_n(\mathbb{R})$ is isomorphic to $\mathbb{R}^\times$.

5 More on Quotient groups:

Let G be a group H be a normal subgroup of G. Consider the quotient group of H in G, $G/H = (\{aH : a \in G\}, *)$, where $\{aH : a \in G\}$ denotes the set of all cosets in G.

5.1 What about the subgroups of G/H? How do they look like?

Take any subgroup K of G such that $H \subseteq K \subseteq G$. Since H is a normal subgroup of G, H is a normal subgroup of K as well. Consider $K/H = \{aH : a \in K\}$. Now the claim is, K/H is a group under the operation $*$ given by $aH * bH = abH$ for all $a, b \in K$.
If $a, b \in K$, $ab \in K$. Thus if $aH, bH \in K/H$, then $abH \in K/H$. So $aH * bH \in K/H$.
$*$ is associative in K/H.
H is the identity element.
$(aH)^{-1} = a^{-1}H \in K/H$, whenever $aH \in K/H$ (if $a \in K$, then $a^{-1} \in K$).
So, K/H is a subgroup of G/H, whenever K is a subgroup of G containing H.

Problem 3 *If you take any subgroup G' of G/H, will it always be of the form K/H for some subgroup K of G containing H?*

Solution: Consider the quotient group G/H and let G' be a subgroup of G/H. Then, $G' = \{aH : a \in G\} \subseteq G/H$.
Is $G' = K/H$ for some K with $H \subseteq K \subseteq G$?
Take $K = \{a \in G : aH \in G'\} \subseteq G$. Now $H \subseteq K$ as for any $h \in H$, $hH = H \in G'$.
If $a \in K, b \in K$, then $ab \in K$.
The operation in K is associative as $K \subseteq G$.
$e_G \in K$, as $H \in G'$
Take any $a \in K$. Then $a^{-1} \in K$, as G' is a group.
Thus K is a subgroup of G containing H and $G' = K/H$.
Thus any subgroup of G/H is of the form K/H, where K is a subgroup of G containing H.