LECTURE 1

09.01.2024

What are algebraic structures?

A collection of objects and some operations over those objects.

Set: a collection of objects

Example:

1. $\{a, b, 1, 2, ☺\}$

2. $\{\{a, b\}, 1, 2\}$

Subsets: A set A is said to be a subset of a set B $(A \subseteq B)$, if every element of A is also an element of B.

Empty set (Φ) A set containing no element.

$\boxed{\text{H.W. Prove that } \Phi \subseteq A \text{ for any set } A.}$

- We can construct new sets using the following operations: union, intersection, complementation, difference, product and others.

Equal sets: Two sets A and B are said to be equal if $A \subseteq B$ and $B \subseteq A$.

How to add structures to a set?

- Let us first understand the concept of relations.

  - Let A be any set. A (binary) relation R, say on A is a subset of $A \times A$.

  - Given A and R, defined over A, we have added the structure R to A.

Some notations:

Let R be a binary relation on a set A. We have:

$$R \subseteq \{(a,b) : a, b \in A\} = A \times A$$

If $(a,b) \in R$, then we say 'a is related to b': aRb

If $(a,b) \notin R$, then we say 'a is not related to b': $a \not R b$.

Examples: 1. $(\mathbb{R}, R)$, where:

- $\mathbb{R}$ is the set of real numbers
- For $a, b \in \mathbb{R}$, aRb iff $a | b$.

2. (A, R), where
- A is any set
- For $a, b \in A$, aRb if $a, b \in A$.

In general we have the following:

Any function is a relation, but not conversely.

Example : (A, R), where
$A = \{1, 2\}$ and,
$R = \{(1,1), (1,2)\}$.

- Let A be a non-empty set. Let R be a relation on A and f be a function on A.

unary R : $R \subseteq A$ — unary f : $f : A \to A$

binary R : $R \subseteq A \times A$ — binary f : $f : A \times A \to A$

⋮

n-ary R : $R \subseteq A^n$ — n-ary f : $f : A^n \to A$.

Note : An n-ary function is an $(n+1)$-ary relation.

Different types of relations :

In what follows, unless otherwise stated, by a relation, we will mean a binary relation.

Let A be a non-empty set and R be a relation on A.

1. **R is reflexive** : for all $a \in A$, $a R a$.

   Examples : 1. $(\mathbb{R}, =)$, $\mathbb{R}$ is the set of real numbers.

   2. $(\mathbb{R}^+, |)$, $\mathbb{R}^+$ is the set of positive real numbers.

2. **R is symmetric** : for all $a, b \in A$, if $a R b$ then $b R a$.

   Examples : 1. $(\mathbb{R}, R)$: $a, b \in \mathbb{R}$ and $a R b$ iff $a - b$ is divisible by 2.

   2. (People, sibling)

3. **R is transitive** : for all $a, b, c \in A$, if

aRb and bRc, then aRc.

Examples: 1. $(\{a, b, c\}, R)$: a, b, c are vertices of a triangle, R is reachability.

2. $(\{a, b\}, R)$: $R = \{(a, a)\}$.

4. R is anti-symmetric: for all $a, b \in A$, aRb and bRa imply $a = b$.

Examples: 1. $(\{a, b\}, R)$, where $R = \{(a, a)\}$

2. $(\mathbb{R}, \leq)$, where $\mathbb{R}$ is the set of real numbers.

5. R is an equivalence relation.

reflexive, symmetric, transitive

6. R is a partial order

reflexive, anti-symmetric, transitive