LECTURE 10

16.02.2024

First Isomorphism Theorem.

Let G and G' be two groups and $f: G \to G'$ be an epimorphism. Then, there is an isomorphism between $G/_{\mathrm{Ker} f}$ and G'.

Proof: Let $H = \mathrm{Ker} f$. Define:

$h: G/_H \to G' : aH \longmapsto f(a)$

We would like to show that

h is an isomorphism.

- $h(aH * bH) = h(abH)$

$$= f(ab)$$

$$= f(a)\, f(b)$$

$$= h(aH)\, h(bH)$$

Thus, h is a homomorphism.

- h is surjective

Let $g' \in G'$. Then, there is $g \in G$, s.t. $f(g) = g'$. Thus, $h(gH) = g'$.

Hence, h is surjective.

- h is injective.

Suppose, $aH, bH \in G/H$, s.t.

$h(aH) = h(bH)$

$\Rightarrow \quad f(a) = f(b)$

$\Rightarrow \quad f(a)(f(b))^{-1} = e_{G'}$

$\Rightarrow \quad f(ab^{-1}) = e_{G'}$

$\Rightarrow \quad ab^{-1} \in H.$

$\Rightarrow \quad aH = bH$ (Check!)

This completes the proof. $\square$

More on quotient groups

Let G be a group and H be a normal subgroup of G. Consider the quotient group of H in G; G/H $= (\{aH : a \in G\}, *)$, where $\{aH : a \in G\}$ denotes the set of all cosets in G.

Now, what about the subgroups of G/H? How do they look like?

- Take any subgroup K of G s.t. $G \supseteq K \supseteq H$. Since H is a normal subgroup of G, H is a normal subgroup of K as well. Consider $K/H = \{aH : a \in K\}$. Now, the claim is K/H is a group under the operation $*$ given by $aH * bH = abH$ for all $a, b \in K$:
  - If $a, b \in K$, $ab \in K$. Thus, if $aH, bH \in K/H$, then $abH \in K/H$. So, $aH * bH \in K/H$.
  - $*$ is associative in K/H.
  - H is the identity element.
  - $(aH)^{-1} = a^{-1}H \in K/H$, whenever $aH \in K/H$ (if $a \in K$, then $a^{-1} \in K$).

So, K/H is a subgroup of G/H, whenever K is a subgroup of G containing H.

Q. If you take any subgroup G' of G/H, will it always be of the form K/H for some subgroup K of G containing H?

Consider the quotient group G/H and let G' be a subgroup of G/H.

Then, $G' = \{aH : a \in G\} \subseteq G/H$,

Is $G' = K/H$ for some K with $G \supseteq K \supseteq H$?

Take $K = \{a \in G : aH \in G'\} \subseteq G$.

- Now, $K \supseteq H$ as for any $h \in H$, $hH = H \in G'$.

- if $a \in K$, $b \in K$, then $ab \in K$.

- the operation in K is associative as $K \subseteq G$.

- $e_G \in K$ as, $H \in G'$.

- Take any $a \in K$. Then, $a^{-1} \in K$, as G' is a group.

Thus, K is a subgroup of G containing H and $G' = K/H$.

Thus any subgroup of G/H is of the form K/H, where K is a subgroup of G containing H.

Products of groups

Let $(G_1, *_1)$ and $(G_2, *_2)$ be two groups. Consider the set $G_1 \times G_2$.

Define a binary operation $*$ on $G_1 \times G_2$, given by:

$$(g_1, g_2) * (g_1', g_2') = (g_1 *_1 g_1', g_2 *_2 g_2'),$$

for all $(g_1, g_2), (g_1', g_2') \in G_1 \times G_2$.

Q. Does $(G_1 \times G_2, *)$ form a group?

A. - $*$ is associative (Check!)

- (e_{G_1}, e_{G_2}) is the identity element (Check!)

- for any $(a, b) \in G_1 \times G_2$, inverse of (a, b), that is, $(a,b)^{-1} = (a^{-1}, b^{-1})$

Yes, it does, and this group is called the product of the groups G_1 and G_2.

H.W. Show that the following are homomorphisms.

① $f_1 : G_1 \to G_1 \times G_2 : g_1 \mapsto (g_1, e_{G_2})$

② $f_2 : G_2 \to G_1 \times G_2 : g_2 \mapsto (e_{G_1}, g_2)$

③ $f_3 : G_1 \times G_2 \to G_1 : (g_1, g_2) \mapsto g_1$

④ $f_4 : G_1 \times G_2 \to G_2 : (g_1, g_2) \mapsto g_2$.

Product of subgroups.

Let G be a group. Can we find subgroups H and K of G such that $H \times K$ is isomorphic to G?

Examples.

1. $G = \{e, a, a^2, a^3\}$

$H = \{e, a^2\}$.

Then, H is a subgroup of G.

Is G isomorphic to $H \times H$?

NO (Why?).

2. $G = \{e, a, a^2, a^3, a^4, a^5\}$.

$H = \{e, b\}$.

$K = \{e, c, c^2\}$

Is G isomorphic to $H \times K$?

Define $f : G \to H \times K$ by

$$a^i \longmapsto (b^i, c^i)$$

f is an isomorphism. (Check!)

Here, one can identify H and K to be subgroups of G (Why?).

Thus, G is isomorphic to the product $H \times K$, where H and K are subgroups of G.

Conjecture: If G is a cyclic group of order n, say, where $n = pq$, with p and q are primes with $p \neq q$, then G is isomorphic to $H \times K$ where H is a cyclic group of order p

and K is a cyclic group of order q.

H.W. Prove this conjecture.

Now, let us generalise this idea to consider any group, not just cyclic groups.

Let G be a group and H and K be subgroups of G. What conditions can we impose on H and K such that G is isomorphic to $H \times K$?

Let us first try to define a map $f: H \times K \to G$. A natural map can be of the form: $f(h,k) = hk$. Now, can we show that this map is an isomorphism? Let's see!

– Is f a homomorphism?

To show: $f((h_1,k_1)\cdot(h_2,k_2)) = f((h_1,k_1))\cdot f((h_2,k_2))$

L.H.S. $= f((h_1h_2, k_1k_2)) = h_1h_2k_1k_2$.

R.H.S $= h_1k_1 \cdot h_2k_2 = h_1k_1h_2k_2$.

To make them equal we need to show that $hk = kh$ for all $h \in H$, $k \in K$. (#)

Under what conditions on H and K can we show # ?

Now, $hk = kh$ would imply $hkh^{-1} \in K$ and $k^{-1}hk \in H$. So, a natural assumption that we can enforce on H and K is that H and K are normal subgroups of G.

Assumption 1: $H \trianglelefteq G$ and $K \trianglelefteq G$.

But, this would not suffice as then we might have: $hk = k'h$ for some $k' \in K$, not necessarily k. So, what more do we need? Let us look into (#):

$hk = kh$, which would imply that:

$hkh^{-1}k^{-1} = e$. So, let us look into

$hkh^{-1}k^{-1}$. Since H and K are normal subgroups, $hkh^{-1}k^{-1} \in H \cap K$. So, if we assume $H \cap K = \{e\}$, we will get (#) for all $h \in H$, $k \in K$.

Assumption 2 : $H \cap K = \{e\}$

Now, with assumptions (1) and (2), we have that f is a homomorphism.

— Let us now try to show that f is injective :

$$f((h_1, k_1)) = f((h_2, k_2))$$

$$\Rightarrow \quad h_1 k_1 = h_2 k_2$$

$$\Rightarrow \quad k_1 = h_1^{-1} h_2 k_2 .$$

$$\Rightarrow \quad k_1 = h_3 k_2 .$$

$$\Rightarrow \quad k_1 k_2^{-1} = h_3 \in H .$$

But, $k_1 k_2^{-1} \in K$. So, $k_1 k_2^{-1} \in H \cap K$.

Hence, $k_1 k_2^{-1} = e$, that is $k_1 = k_2$.

Similarly, $h_1 = h_2$. So, $(h_1, k_1) = (h_2, k_2)$.

- Finally, we have to show that f is surjective:

Let $g \in G$. We have to find $h \in H$ and $k \in K$, s.t. $f((h,k)) = g$, that is $hk = g$. So, we need to show that any $g \in G$ is of the form hk, for some $h \in H$ and $k \in K$. The following assumption would guarantee this.

Assumption (3): $G = HK$.

Thus, we have the following theorem:

Theorem: Let G be a group and H and K be subgroups of G. If

(1) $H \trianglelefteq G$ and $K \trianglelefteq G$,

(2) $H \cap K = \{e\}$, and,

(3) $G = HK$.

then, G is isomorphic to $H \times K$.