LECTURE 15

19.03.2024

Proof of the theorem:

Let R be an integral domain.

Consider $S = R \times R \setminus \{0_R\}$

Define a relation $\sim$ on S as follows:

$(a,b) \sim (c,d)$ iff $ad = bc$.

Is $\sim$ an equivalence relation?

- $\sim$ is reflexive (trivial)
- $\sim$ is symmetric (trivial)
- $\sim$ is transitive: Suppose $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$. To show $(a,b) \sim (e,f)$.

Now, $ad = bc$ and $cf = de$.

$adcf = bcde$.

Then, $adcf - bcde = 0$.

Then, $cd(af - be) = 0$

Then, $cd = 0$ or, $af - be = 0$

- If $cd = 0$, then $c = 0$ as, $d \neq 0$.
  Then, $a = c = e = 0$ and so, $(a,b) \sim (e,f)$.
- If $cd \neq 0$, then $af - be = 0$,
  that is, $af = be$. So, $(a,b) \sim (e,f)$.

Thus $\sim$ is an equivalence relation.

Then S can be partitioned into equivalence classes. Let a/b denote the equivalence class containing (a,b), that is, $a/b = [(a,b)]$.

Let $F = S/\sim = \left\{ a/b : (a,b) \in R \times R \setminus \{0\} \right\}$.

Define $+$ and $\cdot$ on F as follows:

- $a/b + c/d = (ad + bc)/bd$
- $a/b \cdot c/d = ac/bd$

We first need to check that $+$ and $\cdot$ are well-defined:

- if $a_1/b_1 = a_2/b_2$ and $c_1/d_1 = c_2/d_2$, we need to show that $a_1/b_1 + c_1/d_1 = a_2/b_2 + c_2/d_2$ and $a_1/b_1 \cdot c_1/d_1 = a_2/b_2 \cdot c_2/d_2$.

H.W. Prove the well-definedness of $+$ and $\cdot$.

Now, we have to check that $(F, +, \cdot)$ forms a field.

- additive identity: $0/r$, $r \in R \setminus \{0\}$.
- multiplicative identity: r/r, $r \in R \setminus \{0\}$.
- additive inverse of r/s: $-r/s$, $r \in R$, $s \in R \setminus \{0\}$.
- multiplicative inverse of r/s: s/r, $r, s \in R \setminus \{0\}$.

H.W. Prove that $(F, +, \cdot)$ forms a field.

So, we have constructed a field F using the integral domain R.

Now, we need to show that R sits inside F, that is, we have to find an injective homomorphism $h : R \to F$.

Define $h : R \to F : r \longmapsto r/1$

<u>h is injective</u> : Take any $a, b \in R$.

Now, $h(a) = h(b)$ implies $a/1 = b/1$.

implies $a \cdot 1 = b \cdot 1$, that is, $a = b$.

<u>h is homomorphism</u> : Take any $a, b \in R$.

Now, $h(a+b) = \dfrac{a+b}{1} = a/1 + b/1$

$= h(a) + h(b)$.

$h(a \cdot b) = \dfrac{a \cdot b}{1} = a/1 \cdot b/1$.

$= h(a) \cdot h(b)$.

Thus, h is an injective homomorphism.

This completes the proof.

F is called a quotient field of R.

Quotient field.

Let R be an integral domain. Then the quotient field Q_R, say, is in fact the smallest field containing R in the sense that if F is any field such that there is an injective homomorphism $f : R \to F$, then there is an injective homomorphism, $f^* : Q_R \to F$. How do we get this f^*?

Consider

$$\begin{tikzcd} R \arrow[rr, "f"] \arrow[dr, "i"'] & & F \\ & Q_R \arrow[ur, "f^*"'] & \end{tikzcd}$$

$f^* \circ i = f$

Given f, and the map h given above, which we denote here by i, we need to find $f^*: Q_R \to F$ which is an injective homomorphism with $f = f^* \circ i$.

Define $f^*: Q_R \to F$ by :

$$f^*\left(a/b\right) = f(a)\left[f(b)\right]^{-1}$$

- Since $b \neq 0$, $f(b) \neq 0$, as f is injective.
  So, $[f(b)]^{-1}$ exists.

- Let us first check whether f^* is well-defined.

  Suppose $a/b = c/d$. To show $f^*(a/b) = f^*(c/d)$.

  Now, we have $ad = bc$.

  So, $f(ad) = f(bc)$

  i.e., $f(a)f(d) = f(b)f(c)$.

  i.e., $f(a)(f(b))^{-1} = f(c)(f(d))^{-1}$.

Thus, f^* is well-defined.

H.W. Prove that f^* is an injective homomorphism.

Now, we need to show that

$$f^* \circ i = f.$$

Take any $r \in R$.

$$
\begin{aligned}
(f^* \circ i)(r) &= f^*(i(r)) \\
&= f^*\left(\frac{r}{1}\right) \\
&= f(r)\,(f(1))^{-1} \\
&= f(r) \quad [\text{Since, } f(1) = 1_F].
\end{aligned}
$$

Thus we have that Q_R, the quotient field of R is the smallest field containing R.

H.W. Let R be a commutative ring with identity and F be a field. Let $f: R \to F$ be an injective homomorphism. Then $f(1_R) = 1_F$. Prove or disprove.