LECTURE 2

12.01.2024

Different types of functions

In what follows, unless otherwise stated, by a function we will mean <u>many function</u>.

Let A and B be two non-empty sets and $f : A \to B$.

1. <u>f is injective</u>: for all $x, y \in A$
   $f(x) = f(y)$ implies $x = y$.

   <u>Example</u>: $f : \mathbb{R} \to \mathbb{R} : x \longmapsto x + 2$

   $f : A \to A : x \longmapsto x$.

2. <u>f is surjective</u>: for all $b \in B$, there is some $a \in A$ such that $f(a) = b$.

   <u>Examples</u>: same as above.

3. f is bijective : injective and surjective

Counting the number of elements in a set.

Finite set : A set A is said to be finite if A is empty or there is a bijection $f : A \to I_n$, for some $n \in \mathbb{N}$, where $I_n = \{1, 2, 3, \ldots, n\}$.

Infinite set : A set A is said to be infinite if it is not finite.

$\mathbb{N}$ (the set of natural numbers) is an infinite set. Prove this by using PoI : (H.W.)

Let us now have a detailed look at PoI.

Principle of Induction (PoI).

Let $S \subseteq \mathbb{N}$ such that the following holds : (i) $0 \in S$

(ii) if $n \in S$, then $n+1 \in S$

Then, $S = \mathbb{N}$.

Now, how to prove PoI ?

Let us first introduce Well-ordering principle for the set of natural numbers.

Well-ordering Principle (WOP)

Any non-empty subset of $\mathbb{N}$ has a least element.

Theorem. PoI iff WOP.

Proof. Let us assume PoI. To prove WOP. Suppose WOP does not hold. So, there is a non-empty subset S

of $\mathbb{N}$, which does not have a least element.
Let us consider $P(n)$ to be the property for all numbers $i = 0, 1, 2, ---, n$, $i \notin S$.
Now, $0 \notin S$. So, $P(0)$ holds.
Assume $P(k)$ holds for $k \in \mathbb{N}$.
Then, $P(k+1)$ holds (Check!)
Then, by PoI, $P(n)$ holds for all n.
Then, S is empty, a contradiction.
Hence, the result, that is WoP holds. This completes the proof.

Conversely, suppose WOP holds. We have to prove PoI. Take $S \subseteq \mathbb{N}$ satisfying the conditions: (i) $0 \in S$ and (ii) $n \in S$ implies $n+1 \in S$. To prove $S = \mathbb{N}$. Suppose not. Then, $\mathbb{N} \setminus S \neq \Phi$. Then, WOP tells us that $\mathbb{N} \setminus S$ has a least element, k, say. Now, $k \neq 0$

as, $0 \in S$. Then, $k-1 \in \mathbb{N}$. Then, $k-1 \notin \mathbb{N} \setminus S$, as k is the least element of $\mathbb{N} \setminus S$. So, $k-1 \in S$. But then, $(k-1)+1 \in S$, that is, $k \in S$, a contradiction. Hence, we have $S = \mathbb{N}$. This completes the proof. $\square$

<u>Principle of Strong Induction (PoSI).</u>

Let $S \subseteq \mathbb{N}$ satisfying the following conditions

(i) $0 \in S$

(ii) if $k \in S$ for all $k \le n$, then $n+1 \in S$.

Then $S = \mathbb{N}$.

H.W. PoI iff PoSI

<u>How to write 'proof by induction'.</u>

We prove the result by applying induction on n (whatever that may be).

- <u>Basis step</u>: We prove the result for the least possible number that is relevant for the statement.

- Induction Hypothesis: We assume that the statement holds for $n = k$.

- Induction Step: We prove that the statement holds for $n = k+1$.

On infinite sets.

Other examples of infinite sets:

$\mathbb{Z}$: the set of integers

$\mathbb{Q}$: the set of rationals

$\mathbb{R}$: the set of reals.

We now have examples of finite sets and infinite sets.

Now, let us ask the following questions:

1. How to describe the notion of 'number of elements' in a set when it is infinite?

2. Can we distinguish two infinite sets in terms of the 'number of elements' present in those sets?

Let us now try to answer these questions.

A discussion on cardinality of sets.

Two sets A and B are said to be <u>equipotent</u> if there is a bijection between A and B.

Let us define a relation R on the collection of sets given as follows:

A R B if A and B are equipotent.

H.W. R is an equivalence relation.

Let us now mention <u>some concepts based on equivalence relations</u>. If R is an equivalence relation on a set A, define $[a] = \{b \in A : aRb\}$ for any $a \in A$

We call $[a]$ to be the equivalence class of a in A with respect to R.

We have the following:

H.W. $\begin{cases} \text{(i) } a \in [a], \text{ for all } a \in A. \\ \text{(ii) For any } a, b \in A, \text{ either } aRb, \text{ or,} \\ \quad [a] \cap [b] = \Phi. \end{cases}$

Partitioning of a set

A collection of subsets $\{A_i : i \in I\}$ is said to partition a set A if:

(i) $\bigcup_{i \in I} A_i = A$ and (ii) for all $i, j \in I$, $i \neq j$,

$A_i \cap A_j = \Phi$.

H.W. $\begin{cases} \text{Prove that any equivalence relation on } A \text{ gives rise to a partition of } A. \\ \text{Prove that any partition of } A \text{ gives rise to an equivalence relation on } A. \end{cases}$