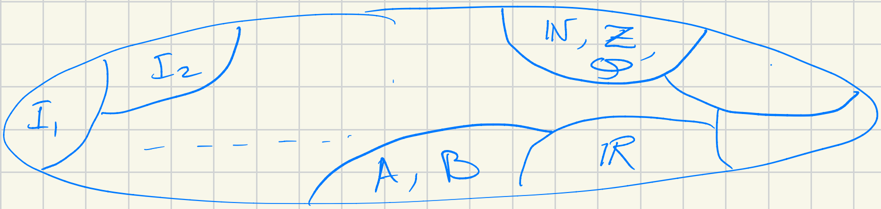
LECTURE 3

16.01.2024

We will continue our discussion on '<u>cardinality of sets</u>'.

From what we discussed last time, we have an <u>equivalence relation</u> on the collection of sets giving us a <u>partition</u> of the collection.



Q. <u>Do $\mathbb{N}$ and $\mathbb{Z}$ belong to the same collection?</u> Yes

Similar questions would arise w.r.to $\mathbb{Z}$ and $\mathbb{Q}$, $\mathbb{Q}$ and $\mathbb{R}$ and others. We will get to this later.

For now, we say that <u>two sets have the same cardinality if there is a bijection between them</u>

Countable sets

A set X is said to be countable if there is a bijection between $\mathbb{N}$ and X.

Uncountable sets

A set X is said to be uncountable if it is not finite and not countable.

Examples of countable sets.

1. $\mathbb{N}$ is countable.

2. Infinite subsets of $\mathbb{N}$.

To show this let us prove a more general result:

Proposition: Take any countable set A. Let E be an infinite subset of A. Then, E is also countable.

Proof. Let A be a countable set.

and $E \subseteq \inf A$. Let us arrange (enumerate / list) the elements of A in a sequence $\{x_n\}_{n \in \mathbb{N}}$ of distinct elements:

$x_1 \quad x_2 \quad x_3 \quad \cdots \quad x_n \quad \cdots$

Let us now construct another sequence $\{n_k\}_{k \in \mathbb{N}}$ as follows:

Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Let n_2 be the smallest positive integer greater than n_1 s.t. $x_{n_2} \in E$. Given $n_1, n_2, \ldots, n_k$, let n_{k+1} be the smallest positive integer greater than n_k, with $x_{n_{k+1}} \in E$.

Now, define a map $f : \mathbb{N} \to E$ by $f(k) = x_{n_k}$. This gives us the required bijection. This completes the proof. $\square$

Note: In some sense, countable sets represent the "smallest infinity".

3. $\mathbb{Z}$ is a countable set.

$$\cdots \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad \cdots$$

$$f : \mathbb{Z} \to \mathbb{N} : \quad f(0) = 0$$
$$f(-n) = 2n$$
$$f(n) = 2n - 1$$

f gives a bijection between $\mathbb{Z}$ and $\mathbb{N}$.

4. $\mathbb{Q}$ is a countable set.

This is a bit more complicated.

Let us consider $\mathbb{Q}^+$ first, the set of all positive rational numbers.

$$\begin{array}{ccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\[1em]
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \cdots \\[1em]
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \cdots \\[1em]
\frac{4}{1} & \frac{4}{2} & \frac{4}{3} & \frac{4}{3} & \cdots \\[1em]
\cdots & & & &
\end{array}$$

$\mathbb{Q}^+ = \left\{ \frac{p}{q} : p \text{ and } q \text{ are prime to each other} \right\}$.

Then, there will not be any problem as all members are distinct.

Another way to deal with such listing is as follows: We collect all the similar representations in the same equivalence class. Then we consider the set of these equivalence classes and provide a bijection.

For example: For the positive rational numbers, $\frac{a}{b} \sim \frac{c}{d}$ iff $ad = bc$.

Now, $\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$. Then, we can prove $\mathbb{Q}$ to be countable if we prove the following.

H.W.

1. Union of two countable sets is countable.
2. Union of a finite set and a countable set is countable.