LECTURE 4

19.01.2024

- Union of finitely many countable sets is countable

Proof idea : It follows by induction using H.W. (1) above..

- Union of countably many countable sets is also countable.

Proof idea : Let $\{A_n : n \in \mathbb{N}\}$ be a countable collection of countable sets. Let $A = \bigcup_{n \in \mathbb{N}} A_n$. To show that A is countable. Let A_n's be pairwise disjoint.

Now, we have that each A_n is countable. Then, each A_n can be written as : $\{a_{n1}, a_{n2}, a_{n3}, \ldots\}$, that is, $A_n = \{a_{nk}\}_{k \in \mathbb{N}}$. Then we have the following arrangement for the members of A:

$$\begin{array}{cccccc}
a_{11} & a_{12} & a_{13} & a_{14} & - & - - - - \\
a_{21} & a_{22} & a_{23} & a_{24} & - & - - - - \\
a_{31} & a_{32} & a_{33} & a_{34} & - & - - - \\
a_{41} & a_{42} & a_{43} & a_{44} & - & - - - - \\
- & - & - & - & - & - - - - - -
\end{array}$$

This enumeration gives a bijection between $\mathbb{N}$ and A. So A is countable.

Products of countable sets.

- Let A and B be two countable sets. Then, $A \times B$ is countable.

Proof idea: $A = \{a_1, a_2, \ldots\}$

$B_n = \{b_1, b_2, \ldots\}$

Then, one can think of the following

$$\begin{array}{cccc}
(a_1, b_1) & (a_1, b_2) & (a_1, b_3) & - - - - - \\
(a_2, b_1) & (a_2, b_2) & (a_2, b_3) & - - - - \\
- & - & - & - - - - - - -
\end{array}$$

- Let $A_1, A_2, \ldots, A_n$ be n countable sets, where $n \geq 1$. Then $A_1 \times A_2 \times \cdots \times A_n$ is also countable.

Proof idea: The proof follows from the previous result using induction.

- Let $A_1, A_2, \ldots$ be a countable collection of countable sets. Then what can we say about $\prod_{n \in \mathbb{N}} A_n$, that is countable product of countable sets?

Let us consider the set $A = \{0, 1\}$.
Let $B = \prod_{n \in \mathbb{N}} A_n$, where $A_n = A$ for all n.

Is B countable? NO

Proof. Suppose B is countable.
Then $B = \{b_1, b_2, b_3, \ldots\}$. Now, each b_i is a sequence of 0's and 1's.
Then, $b_i = (b_{i1}, b_{i2}, b_{i3}, \ldots)$, where

$b_{ij} \in \{0, 1\}$ for all $j \in \mathbb{N}$.

Let c be a sequence of 0's and 1's defined as follows:

$$c_i = \begin{cases} 0 & \text{if } b_{ii} = 1 \\ 1 & \text{if } b_{ii} = 0 \end{cases}, \quad i \in \mathbb{N}.$$

Then, $c \neq b_i$ for any $i \in \mathbb{N}$. Thus, $c \notin B$, a contradiction. Hence, our assumption that B is countable cannot be true. Hence, B is uncountable. This completes the proof. $\square$

What went on in this proof?

$$b_1 = \underline{b_{11}}\ b_{12}\ b_{13}\ b_{14} \ \text{— — — —}$$

$$b_2 = b_{21}\ \underline{b_{22}}\ b_{23}\ b_{24} \ \text{— — — —}$$

$$b_3 = b_{31}\ b_{32}\ \underline{b_{33}}\ b_{34} \ \text{— — — — —}$$

— — — — — — — — — —

$$b_i = b_{i1}\ b_{i2}\ b_{i3}\ b_{i4} \ \text{— — — — —}$$

— — — — — — — — — — —

To construct c, we considered b_{ii}'s. This is what is famously known as '<u>diagonalization argument</u>'.

H.W. $\mathbb{R}$ is uncountable

Some notations:

When we have $\prod_{n\in\mathbb{N}} A_n$, where $A_n = A$ for all n; we denote $\prod_{n\in\mathbb{N}} A_n$ by $A^{\mathbb{N}}$;

any tuple over A which is countable in size can be represented by a function $f : \mathbb{N} \to A$. [X^Y denotes the set of all functions from Y to X.]

This not only holds for $\mathbb{N}$, but for any indexing set I. For example, if we consider the collection $\{A_i : i \in I\}$, $A_i = A$ for all $i \in I$, $\prod_{i\in I} A_i$ is also given by A^I.

$$\begin{cases} 2^{\mathbb{N}} \text{ is uncountable } (2 = \{0,1\}) \\ \mathbb{R} \text{ is uncountable.} \end{cases}$$

H.W. Prove that there is a bijection between $2^{\mathbb{N}}$ and $\mathbb{R}$.

How many 'infinites' do we have?

- Countable sets (can be represented by $\mathbb{N}$)
- Uncountable sets (do all uncountable sets have bijections between each other? NO

Some definitions

- Let A be any set. We denote the cardinality of A by $|A|$. For example, we have $|\mathbb{N}|$ denoting the cardinality of $\mathbb{N}$ and similarly we have $|\mathbb{R}|$. Let $|\mathbb{N}| = \aleph_0$, and let $|\mathbb{R}| = c$ [We have $2^{\aleph_0} = c$].
- If there is an injection from a set A to a set B, then we

denote it by : $|A| \leq |B|$. And, $<$ denote the strict order.

Schröder - Bernstein Theorem : Let A and B be two sets such that there is an injection from A to B and an injection from B to A. Then, there is a bijection between A and B.

Finally, Cantor showed us that 'there are infinitely many infinities'

Let A be any set. Then $|A| < |\wp(A)|$

Proof : We have an injection from A to $\wp(A)$ given by $f : A \to \wp(A)$: $a \mapsto \{a\}$. It is enough to show that there cannot be a surjection from A to $\wp(A)$. Suppose not. Then, let $g : A \to \wp(A)$ be a surjection.

Let us consider the following set:

$B = \{ b \in A \mid b \notin g(b) \}$. Then, $B \subseteq A$, so $B \in \wp(A)$. Now, g is surjective. So, there is $a \in A$, such that $g(a) = B$.

Does $a \in g(a)$ or, $a \notin g(a)$?

Suppose $a \in g(a) = B$. Then: $a \notin g(a)$.

Suppose $a \notin g(a) = B$. Then: $a \in g(a)$.

So, we arrive at a contradiction. Hence, our original assumption about the existence of a surjection is not true. Thus, there cannot be a surjection from A to $\wp(A)$.

So, $|A| < |\wp(A)|$. This completes the proof. $\square$