LECTURE 7

06.02.2024

Kernel of a homomorphism.

Let $f : G \to G'$ be a homomorphism.
Take $g \in G$ and $h \in \operatorname{Ker} f$
Consider ghg^{-1} in G.

$$\text{Now } f(ghg^{-1}) = f(g)\, f(h)\, f(g^{-1})$$
$$= f(g)\, f(g^{-1})$$
$$= e_{G'}$$

Then, $ghg^{-1} \in \operatorname{Ker} f$.

So $\underline{g \operatorname{Ker} f\, g^{-1} \subseteq \operatorname{Ker} f}$

Is $\underline{\operatorname{Ker} f \subseteq g \operatorname{Ker} f\, g^{-1}}$? YES!!

H.W. Prove that $\operatorname{Ker} f \subseteq g \operatorname{Ker} f\, g^{-1}$ for any $g \in G$.

We have, $\underline{g \operatorname{Ker} f\, g^{-1} = \operatorname{Ker} f}$ for all $g \in G$.

We say: Ker f is a <u>normal subgroup</u> of G.

Normal subgroup

Let G be a group and H is a subgroup of G. H is said to be a normal subgroup of G if $gHg^{-1} = H$ for all $g \in G$. We denote by $H \trianglelefteq G$.

$\{ghg^{-1} : h \in H\}$

H.W. Prove that gHg^{-1} is a subgroup of G, where H is a subgroup of G and $g \in G$.

Example

① For any homomorphism f on a group G, $\mathrm{Ker}\, f$ is a normal subgroup of G.

② If G is a commutative group, then any subgroup of G is a normal subgroup (Why?)

Q. What about a subgroup of a group is not a normal subgroup?

Let us consider SG_3, a non-commutative group, and the subgroup: $\{e, \tau\}$.

$e : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ $\tau : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$

Consider $\tau' : \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

Now, $\tau' \tau \tau'(3) = 2$

This shows that $\{e, \tau\}$ is not a normal subgroup of SG_3.

Q. Is there any non-trivial normal subgroup of SG_3?

H.W. Show that $\{e, \sigma, \sigma'\}$ is a normal subgroup of SG_3. Here,

$\sigma : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $\sigma' : \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

H.W. Let G and G' be two groups and $f : G \to G'$ be a homomorphism. Prove that f is a monomorphism iff $\operatorname{Ker} f = \{e_G\}$.

More examples.

(1) Consider $f : \mathbb{Z} \to SG_2$: $\{1,2\}$

$$f(z) = \begin{cases} e & \text{if } z \text{ is even} \\ \tau & \text{if } z \text{ is odd.} \end{cases}$$

$\operatorname{Ker} f = 2\mathbb{Z}$.

(2) Consider $f : GL_n(\mathbb{R}) \to GL_1(\mathbb{R})$:

$f(A) = \det A$.

$\operatorname{Ker} f = \{A \in GL_n(\mathbb{R}) : \det A = 1\}$

We denote this set by $SL_n(\mathbb{R})$, termed as 'Special linear group of order n'.

Note that $SL_n(\mathbb{R})$ is a normal subgroup of $GL_n(\mathbb{R})$.

Next, we will explore a connection between SG_n and $GL_n(\mathbb{R})$.

Consider SG_3 and $\tau \in SG_3$.

$$\tau : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

Consider $A_\tau = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Take $\sigma : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

Then $A_\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

So, the basic idea of a permutation matrix corresponding to a permutation $\sigma \in SG_n$, say, is:

$$\left(\begin{array}{c} \text{The } j\text{th column} \\ \text{in } I_n \text{ is replaced} \\ \text{by the } \sigma(j)\text{th column} \end{array} \right)$$

Determinant of any such permutation matrix is 1 or -1.

H.W. Prove the statement above.

So we have a map g : $\det \circ f$

$$g : SG_n \xrightarrow{f} GL_n(\mathbb{R}) \xrightarrow{\det} GL_1(\mathbb{R})$$

$$\operatorname{Ker} g = \{\sigma : \det(A_\sigma) = 1\}.$$

Now, Ker g forms a normal subgroup of SG_n. This group is denoted by AG_n (Alternating group). These ideas give rise to the following concepts :

Even permutation : A permutation σ such that A_σ is obtained from I_n by an even number of exchanges.

<u>Odd permutation</u> : A permutation σ such that A_σ is obtained I_n by an odd number of exchanges.

<u>Another way to consider these symmetric or permutation groups.</u>

SG_2 : $\overset{e}{\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}}$ $\overset{\tau}{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}$

SG_3 : $\overset{e}{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}}$ $\overset{\tau}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}}$ $\overset{\tau'}{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}}$

$\overset{\tau''}{\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}}$ $\overset{\sigma}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}}$ $\overset{\sigma'}{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}}$

In SG_2 : $\tau = (1\ 2)$

In SG_3 : $\tau = (1\ 2)$, $\tau' = (1\ 3)$, $\tau'' = (2\ 3)$

$\sigma = (1\ 2\ 3) = (1\ 3)(1\ 2)$

$\sigma' = (1\ 3\ 2) = (1\ 2)(1\ 3)$

Here, we are considering cycles and transpositions (cycles of length 2) to represent these permutations or bijections.

An example

Consider the following permutation.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 8 & 5 & 6 & 4 & 7 & 1 \end{pmatrix}$$

Is it even or odd?

$$\sigma = (1\ 2\ 3\ 8) \circ (4\ 5\ 6)$$
$$= (1\ 8) \circ (1\ 3) \circ (1\ 2) \circ (4\ 6) \circ (4\ 5).$$

It is an odd permutation.

Coming back to our discussion on kernels of homomorphisms, we have $\operatorname{Ker} f \trianglelefteq G$ where f is a homomorphism on G. Let us now ask the following: Take $H \trianglelefteq G$. Is there an f such that $\operatorname{Ker} f = H$?

Let us first consider an example.

Let G be a group. Consider $Z(G)$, the center of a G defined by:

$$Z(G) = \{ h \in G : hg = gh \text{ for all } g \in G \}$$

H. W. Prove that $Z(G) \trianglelefteq G$.

Q. Is there a homomorphism f with domain G, s.t. $\operatorname{Ker} f = Z(G)$?

Let us first introduce a group

$$\operatorname{Aut} G = \{ f \mid f : G \to G \text{ is an isomorphism, } f \text{ is an automorphism on } G \}$$

H. W. Prove that $\operatorname{Aut} G$ forms a group.