LECTURE 8

09.02.2024

We would like to find a map $f : G \to \text{Aut } G$ s.t. $\text{Ker } f = Z(G)$.

Note that for any $g \in G$, $f(g)$ is a member of Aut G, that is, a map from G to G, which is an isomorphism.

How to define $f(g)$?

Define $f(g) : G \to G$ as follows:

$$f(g)(h) = g h g^{-1}$$

Claim : $f(g) \in \text{Aut } G$.

To prove this claim, we show:

(1) $f(g)(h \cdot h') = f(g)(h) \cdot f(g)(h')$

(2) $f(g)$ is a bijection.

Proof of (1) : $f(g)(h \cdot h')$

$$= g(h \cdot h')g^{-1}$$

$$= (gh)(h'g^{-1})$$

$$= (gh)(g^{-1}g)(h'g^{-1})$$

$= (ghg^{-1})(gh'g^{-1})$

$= f(g)(h) \cdot f(g)(h')$.

Proof of (2): - $f(g)$ is an injection.

Let $h_1, h_2 \in G$ s.t. $f(g)(h_1) = f(g)(h_2)$.

Then, $gh_1g^{-1} = gh_2g^{-1} \Rightarrow h_1 = h_2$.

- $f(g)$ is surjective.

Take any $g' \in G$. To find $h \in G$ s.t. $f(g)(h) = g'$ or, $ghg^{-1} = g'$

or, $h = g^{-1}g'g$.

Thus: $f(g) \in \mathrm{Aut}\, G$.

Now, Ker f $= \{g \in G : f(g)$ is the identity map$\}$

$= \{g \in G : f(g)(h) = h$ for all $h \in G\}$

$= \{g \in G : ghg^{-1} = h$ for all $h \in G\}$

$= \{g \in G : gh = hg$ for all $h \in G\}$

$= Z(G)$.

Maps and equivalence relations:

Let S and T be two non-empty sets. Let $f: S \to T$. Define a binary relation $\sim$ on S given by $a \sim b$ iff $f(a) = f(b)$ for every a, b in S. Then it can be shown that $\sim$ is an equivalence relation.

What more can we say when these maps are group homomorphisms?

Let G and G' be two groups and $f: G \to G'$ be a homomorphism. Now, let us consider $\operatorname{Ker} f = H$, say. Also, let $\mathcal{K}$ be the set of all equivalence classes induced by the map f.

We have $H \in \mathcal{K}$.

Proposition: Any member of $\mathcal{K}$ is of the form $aH = \{ah : h \in H\}$, $a \in G$.

Proof: Take any $a \in G$. We need to show that if $b \sim a$, then $b \in aH$. and if $b \in aH$ then $b \sim a$, for any $b \in G$.

- Suppose $a \sim b$. Then: $f(a) = f(b)$.
  that is, $\left(f(a)\right)^{-1} f(b) = e_{G'}$, that is,
  $f(a^{-1}b) = e_{G'}$, that is, $a^{-1}b \in H$, that is,
  $a^{-1}b = h$ for some h in H. Thus,
  $b = ah \in aH$.
- Suppose $b \in aH$. Then, $b = ah$ for some $h \in H$. Then, $f(b) = f(ah) = f(a) \cdot f(h) = f(a) \cdot e_{G'}$
  $= f(a)$. Hence, $b \sim a$.

This completes the proof. $\square$

Proposition: For any $a \in G$, H and aH have the same cardinality, that is, there exists a bijection between H and aH.

Proof. Let us define $g: H \to aH$ by:
$g(h) = ah$. We first show that g is an injection. Let $h_1, h_2 \in H$.
Now, $g(h_1) = g(h_2)$
$\Rightarrow ah_1 = ah_2$

$\Rightarrow \quad h_1 = h_2$.

We now show that g is a surjection.
Take any $b \in aH$. Then, $b = ah$ for some $h \in H$. So, we have $g(h) = ah = b$.
This completes the proof. $\square$.

Corollary: If G is a finite group, then, $|G| = |\operatorname{Ker} f| \cdot |\operatorname{Image}(f)|$

An application of this corollary:

Consider SG_n and AG_n.

We have $|SG_n| = n!$

Consider $g : SG_n \to \{1, -1\}$ (via $GL_n(\mathbb{R})$)

$\operatorname{Ker} g = AG_n$.

So, $|SG_n| = |AG_n| \cdot 2$.

i.e. $|AG_n| = n!/2$

Now, we would generalize this discussion with respect to any subgroup.

Cosets.

Let G be a group and H be a subgroup of G. A left coset of H in G is a set $\{ah : h \in H\}$ for some $a \in G$. We denote this set by aH. A right coset of H in G is a set $\{ha : h \in H\}$ for some $a \in G$. We denote this set by Ha.

In what follows, we will denote 'left cosets' as 'cosets', unless otherwise specified.

We have seen that by considering the subgroup $\text{Ker} f$ corresponding to a homomorphism f with domain G, we found a partition of G in terms of cosets of $\text{Ker} f$ in G.

Can we get this result for any subgroup H of G?

<u>Proposition</u>: Let K denote the set of all cosets of H in G. Then, G can be partitioned by members of K, that is:

(1) $G = \bigcup_{a \in G} aH$

(2) for any $aH, bH \in K$, either, $aH = bH$ or, $aH \cap bH = \Phi$.

<u>Proof</u>: (1) Take any $g \in G$. To show that $g \in aH$ for some $a \in G$. Now, $g \in gH$ and we are done.

(2) Take any $aH, bH \in K$. If $aH = bH$, we are done. Suppose not, that is $aH \neq bH$. We have to show that $aH \cap bH = \Phi$. Suppose not.

Let $c \in aH \cap bH$. Then $c \in aH$ and $c \in bH$.
So, $c = ah_1 = bh_2$ for some $h_1, h_2 \in H$.
Then, $a = bh_2h_1^{-1} \in bH$. So, $aH \subseteq bH$.
Similarly, we can show that $bH \subseteq aH$.
So, $aH = bH$, a contradiction. Hence, the proof is complete. $\square$

Thus, G can be partitioned by the cosets of H in G, where H is a subgroup of G. Also, the cardinality of each such coset of H is the same as that of H.

Let us denote the number of such cosets of H in G to be the index of H in G, written as $[G:H]$.

Corollary: If G is a finite group

then $|G| = |H| \cdot [G:H]$.

This immediately tells us that for a finite group G, and a subgroup H of G:

Lagrange's Theorem: The order of a subgroup of a finite group G divides the order of G.

Applications of Lagrange's Theorem.

(1) Let G be a finite group and let $g \in G$. Then, $O_G(g) \mid |G|$.

Proof: Take $g \in G$. Consider $\langle g \rangle$, the cyclic subgroup of G generated by g. Now, by Lagrange's theorem, $|\langle g \rangle| \mid |G|$. But $|\langle g \rangle| = O_G(g)$. Hence, the result. $\square$