Project Documentation

Course: Logic For Computer Science

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Contents

1 Introduction:-

The union of two branches such as Group theory and Graph theory has been known since 1878 when Cayley's graph was first defined. In the following years, various graphs have been associated with groups. In 1955, Brauer and Fowler introduced the commuting graph, in early 2000 the directed power graphs were related to semigroups in, and only in 2009, Chakrabarty, Gosh, and Sen defined the undirected power graph, which we will call the power graph. Approaching a group through a graph associated with it allows one to focus on some specific properties of the group. In this presentation, we will focus on directed power graphs and power graphs defined on finite groups. So, the rest of this documentation is organized as:- In section 2 basic definitions of groups and graphs are recalled here. In section 3 we have introduced the power graph of a group. In this section, we have given some propositions with its proof. In section 4 we have introduced another counterpart of the power graph which is the directed power graph. In the last section 5 we have concluded the whole documentation with some open questions.

2 Basic Definition:-

In this section we will recall some basic definitions of groups and graphs.

2.1 Recalls of Group:-

Definition 1. *A **Group** is a non-empty set G with an operation* $* : G \times G \to G$*, satisfying the following properties:*

(i). $*$ *is associative.*

(ii). There exist an element $e \in G$*, such that* $g * e = e * g = g$ *for all* $x \in G$*.*

(iii). For each $g \in G$*, there is g' such that* $g * g' = g' * g = e$*.*

Example:-

- $(\mathbb{Z}, +)$ forms a group in which 0 is the identity element and -a is the inverse a $\in \mathbb{Z}$.

In this presentation, the groups we have considered are all finite. Let G be a group. The identity element of G, will be simply denoted with 1. Let H be a non-empty subset of G, then

Definition 2. *(H, $*$) is said to be a sub-group of (G, $*$) if (H, $*$) forms a group itself.*

To show that (H, $*$) forms a subgroup of (G, $*$) we have to check :

- for all a, b $\in$ H, a$*$b $\in$ H.

- H has as identity element, e_H, say.
- every element $h \in H$ has an inverse $h^{-1} \in H$.

Example:-

- (m$\mathbb{Z}$, +) is a subgroup of ($\mathbb{Z}$, +).

So, order of a group is nothing the cardinality of the G, i.e., $|G|$. Now, if H is a subgroup of G, then we write H $\leq$ G, and if H is a proper subgroup, then we write H < G.

Definition 3. *G is called a cyclic group if there exists $g \in G$ such that $G = \langle g \rangle$.*

Example:-

- SG_3 is an example of a cyclic group.

Let (G, $*$) be a group and let g $\in$ G with g $\neq e_G$. **Order of g** is the least positive integer m such that $g^m = e_G$. If no such m exists then we say that the order of g is infinite. Let $O_G(g)$ denote the order of g in G. Hence, the order of the O(τ) = O(τ') = O(τ") = 2 and O(σ) = O(σ') = 3.

2.2 Recalls of Graph:-

Definition 4. *We define a graph Γ as a couple Γ = (V_Γ, E_Γ) where V_Γ is a not empty finite set of elements called vertices and E_Γ is a subset of ($V_\Gamma \times V_\Gamma$).*

If e = {x, y} for some distinct x and y in V_Γ, we also say that such elements, x and y, are joined or adjacent. The elements of E_Γ are called edges.

Definition 5. *We define a directed graph $\vec{\Gamma}$, or digraph, as a couple $\vec{\Gamma}$ = ($V_{\vec{\Gamma}}, A_{\vec{\Gamma}}$) where $V_{\vec{\Gamma}}$ is a not empty finite set of elements called vertices and $A_{\vec{\Gamma}}$, whose elements are called arcs, is a subset of $(V_{\vec{\Gamma}} \times V_{\vec{\Gamma}}) \setminus \Delta$ where Δ = {(x, x) | $x \in V_{\vec{\Gamma}}$.*

We say that, if (x, y) or (y, x) is in $A_{\vec{\Gamma}}$, then x and y are joined. We also say that a = (x, y) $\in A_{\vec{\Gamma}}$ is directed from x to y and also that a has direction from x to y. If (x, y) $\in A_{\vec{\Gamma}}$, then it said x dominates y. The cardinality of the vertex set of both graphs and digraphs is called the order of the graph/digraph.

Definition 6. *A digraph G is **transitive** if given x, y, z $\in$ V such that (x, y), (y, z) $\in$ E(G), then we have that (x, z) $\in$ E(G).*

Definition 7. *Given a graph Γ we call **subgraph of** Γ a graph Γ' with $V_{\Gamma'} \subseteq V_\Gamma$ and $E_{\Gamma'} \subseteq E_\Gamma$.*

Definition 8. *A subgraph Γ' is **induced** by his vertex set if, for all x, y $\in V_{\Gamma'}$, {x, y} $\in E_{\Gamma'}$ E if and only if {x; y} $\in E_\Gamma$.*

Definition 9. *A graph is called as a **complete graph** of n vertices, denoted by K_n, if {x; y} $\in E_{K_n}$ for all x, y $\in V_{K_n}$.*

We call complete digraph of n vertices, denoted by $\vec{K}_n$, the digraph such that $A_{\vec{K}_n} = (V_{\vec{K}_n} \times V_{\vec{K}_n}) \setminus \Delta$.

3 Power Graph:-

In this section we will introduce the power graphs. Let us first define a power graph.

Definition 10. *Let G be a group. The power graph of G, denoted as $\mathcal{P}(G)$, is defined by $V_{\mathcal{P}}(G)$ = G and e = {x, y} $\in \mathcal{P}(G)$ if $x \neq y$ and there exists a positive integer m such that $x = y^m$ or $y = x^m$.*

Now, since, in a group G, every element has a finite order, therefore, in $\mathcal{P}(G)$, the identity element is a star vertex. Now, let us look at an example. In figure

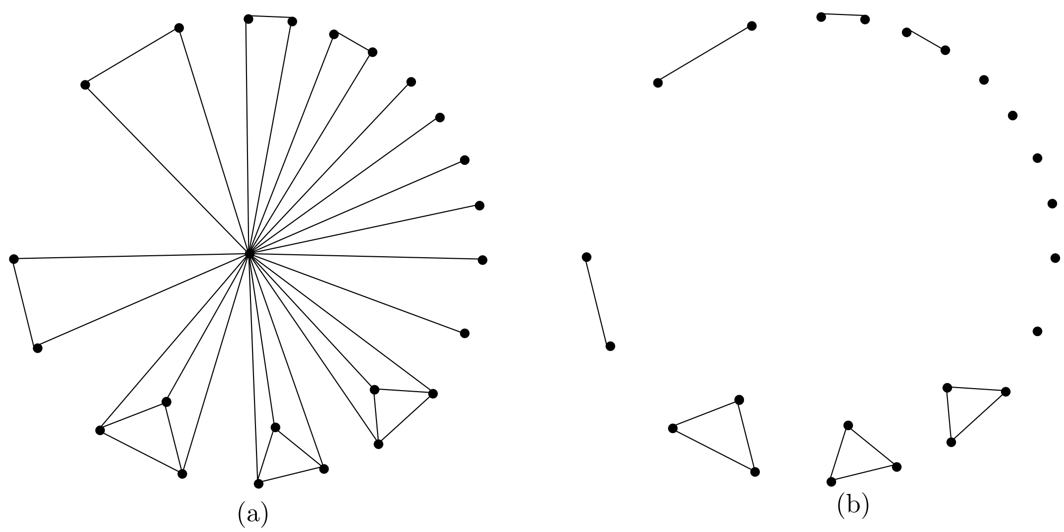

Figure 1: Power graph of S_4 or $\mathcal{P}(S_4)$

1(a) there is a representation of $\mathcal{P}(S_4)$. From the figure, we can recognize only the star vertex as the identity element of S_4. One observation we can see that:- **for all x $\in V_{\mathcal{P}}(G)$, we have that deg(x) $\geq$ o(x) - 1**. This holds because, given x $\in V_{\mathcal{P}}(G)$, {x, y} $\in E_{\mathcal{P}}(G)$ for all y $\in \langle x \rangle \setminus \{x\}$ and we have $|\langle x \rangle|$ = o(x).

Definition 11. *A graph is called the proper power graph of G, which is denoted by $\mathcal{P}^*(G)$ if $\mathcal{P}^*(G)$ is an induced subgraph by $G\setminus \{1\}$.*

In figure 1(b) we see how $\mathcal{P}^*(S_4)$ appears.

Proposition 1. *Let G be a group and $H \leq G$. Then $\mathcal{P}(G)_H = \mathcal{P}(H)$.*

Proof. By definition of power graph and induced subgraph we immediately have $V_{\mathcal{P}}(H)$ = H = $V_{\mathcal{P}(G)_H}$. Now e = {x, y} $\in E_{\mathcal{P}(H)}$ if and only if e $\in E_{\mathcal{P}(G)_H}$ since, by definitions, both are equivalent to x $\neq$ y, x, y $\in$ H and there exists a positive integer m such that x = y^m or y = x^m. Thus $\mathcal{P}(H) = \mathcal{P}(G)_H$. □

Proposition 2. *Let G be a cyclic group. If G is not a p-group, then for all x $\in G \setminus \{1\}$ with $\langle x\rangle \neq= G$ there exists $y \in G$ such that x and y are not joined in $\mathcal{P}(G)$.*

Proof. Let G = $\langle g\rangle$ and let o(g) = $p_1^{a_1}, p_2^{a_2}, \cdots, p_k^{a_k}$, k for an integer k $\geq$ 2, p_1, p_2; $\cdots$, p_k distinct primes and $a_i \in$ N for all i $\in$ [k]. Now let x $\in$ G that satisfies the hypothesis. Then, since $o(x)|o(g)$, o(x) = $p_1^{b_1}, p_2^{b2}, \cdots, p_k^{b_k}$ with, for all i $\in$ [k], b_i an integer such that $0 \leq b_i \leq a_i$. Thus, as $\langle x\rangle \neq$ G, there is an index i such that $b_i < a_i$. Take y $\in$ G with o(y) = $p_i^{a_i}$ then $o(y)|o(x)$ and $o(x)|o(y)$. Therefore y is not joined to x in $\mathcal{P}(G)$. □

Corollary 1. *Let G be a cyclic group. Then $\mathcal{P}(G)$ is a complete graph if and only if G is cyclic of prime power order.*

Proof. Assume that $G \cong C_{p^n}$ for some prime p and n $\in \mathbb{N}_0$. We proceed by induction on the order of G. **Base Step:-** For $|G|$ = 1 and for $|G|$ = p we have that $\mathcal{P}$(G) is a complete graph since G contains only the identity and the generators of the group, that are joined to all others elements. Now let $|G| = p^n$ for some $n \in \mathbb{N}$. Surely there exists y $\in$ G with order p^{n-1}.

inductive hypothesis:-, P($\langle y\rangle$) is complete. Therefore consider two elements x, z $\in$ G. If one of them is a generator, then they are joined. Otherwise, since the elements of $G \setminus \langle y\rangle$ are exactly the generators of G, we have that x; z $\in \langle y\rangle$, and hence they are joined. Thus $\mathcal{P}$(G) is a complete graph.

Assume next that $\mathcal{P}(G)$ is complete. By Proposition 2, since G is cyclic, it must be of prime power order otherwise there exist at least two vertices that are not joined. □

4 Directed Power Graph:-

Let's abandon for a moment the power graph to talk about its directed counterpart. Let us now introduce another type of power graph, that is the directed power graph.

Definition 12. *Let G be a group. The directed power graph of G, denoted $V_{\vec{\mathcal{P}}(G)}$, is defined by $V_{\vec{\mathcal{P}}(G)} = G$ and (x, y) $\in A_{\vec{\mathcal{P}}(G)}$ if $x \neq y$ and there exists a positive integer m such that $y = x^m$.*

For an example of such a directed power graph, take a look at at figure2 where the group is the dihedral group of 8 elements:

$$D_4 = \{1, a, a^2, a^3, bab, a^2b, a^3b\}$$

and

$$A_{\vec{\mathcal{P}}(D_4)} = \{(a,1), (a^2,1), (a^3,1), (b,1), (ab,1), (a^2b,1), (a^3b,1), (a,a^3), (a^3,a), (a,a^2), (a^2,a), (a^2,a^3)\}$$

from now onwards we always refer to $A_{\vec{\mathcal{P}}(G)}$ as A.

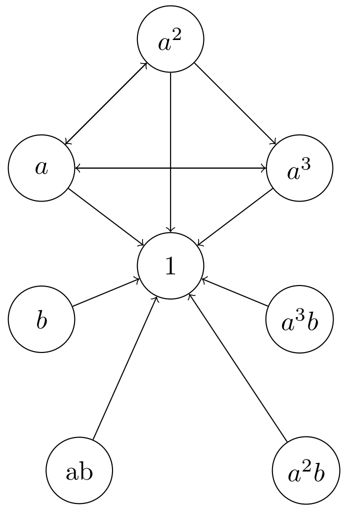

Figure 2: The directed power graph of D_4

Proposition 3. *Let G be a group. Then $\vec{\mathcal{P}}(G)$ is transitive.*

Proof. Let x, y, z ∈ G such that (x, y), (y, z) ∈ A. Then there exist two integers m and n such that y = x^m and z = y^n. Hence we have z = x^{mn}, it follows that (x, z) ∈ A holds. □

Remark:- Let G be a group. Then {x, y} ∈ E if and only if (x, y) ∈ E or (y, x) ∈ E. The Remark above tells us that we can present E if we know A. Indeed $\vec{\mathcal{P}}(G)$ shows us which elements of G are joined in $\mathcal{P}(G)$.

5 Conclusion:-

So, in this documentation we have studied the connection between groups and graphs. Mostly here we have studied the cyclic group and how we can visualize the groups through graphs. Moreover, in this area there are some open problems:

- Does there exist a formula for the maximum length of a cycle/path of a power graph?
- What can be said about a group with a fixed maximum length of a cycle/path of its power graph?