POWER GRAPH OF FINITE GROUPS

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AGENDA

Introduction

Basic Definition

Power Graph

Proper Power Graph

Directed Power Graph

Conclusion

INTRODUCTION

- Contamination between branches of mathematics is interesting
- Union of two branches such as Group theory and Graph theory
- 1878 when Cayley's graph was defined
- In 1955, Brauer and Fowler introduced
    - commuting graph
- In early 2000 the directed power graphs was introduce.

GROUP

- Group:- G is non-empty , an operation $* : G \times G \to G$
  - $*$ is associative
  - e_G identity element
  - a^{-1} is inverse for all a belongs to G.
- Example:- $(\mathbb{Z}, +)$
- All group considered here are finite.
- Order of a group is $|G|$.
- Sub-group:-
  - $H \subseteq G$
  - $(H, *)$ is a group itself.
- Example: - $(m\mathbb{Z}, +)$ is a sub-group of $(\mathbb{Z}, +)$
  - $\mathbb{Z}$ is the set of integers.
- Cyclic Group : -
  - g belongs to G
  - $G = \langle g \rangle$
  - G is known as generator

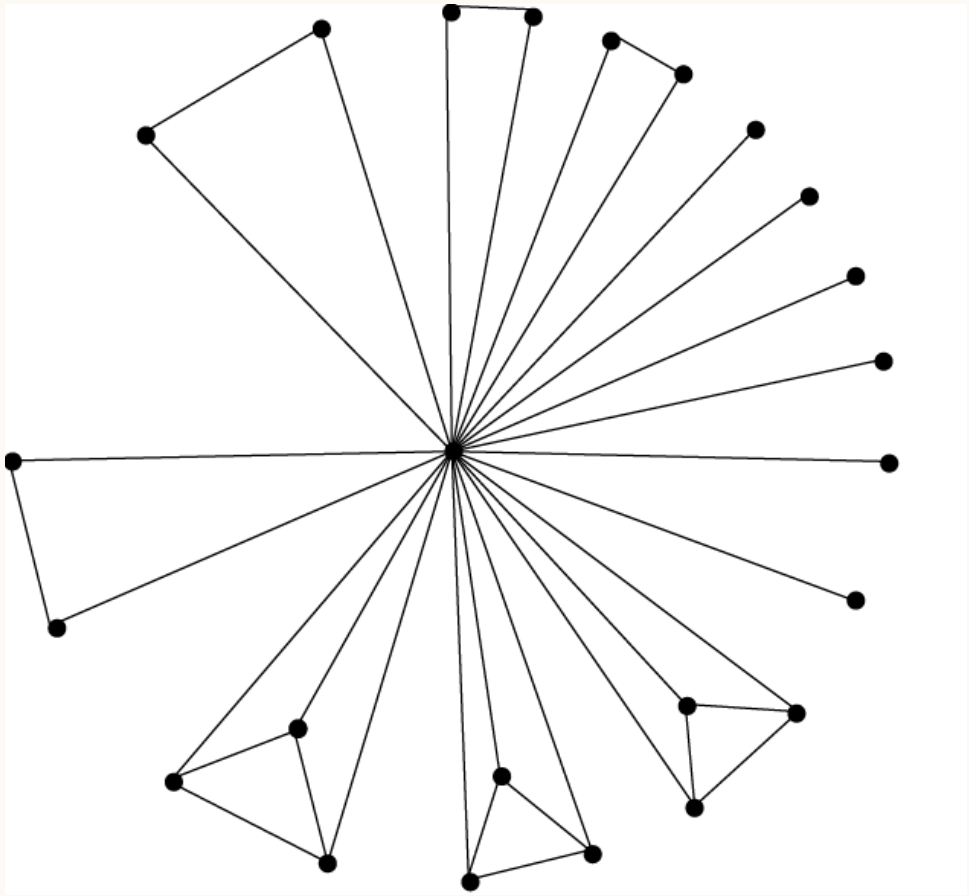
GRAPH

- Graph :- a couple $\Gamma = (V_\Gamma, E_\Gamma)$
  - V_Γ set of vertices
  - E_Γ edge set
  - $E_\Gamma \subseteq V_\Gamma \times V_\Gamma$
- $e = \{x, y\}$
  - for some distinct x and y in V_Γ
  - Adjacent
- Directed Graph:- a couple $\vec{\Gamma} = (V_{\vec{\Gamma}}, E_{\vec{\Gamma}})$
  - $V_{\vec{\Gamma}}$ set of vertices
  - $E_{\vec{\Gamma}}$ arc set
  - $E_{\vec{\Gamma}} \subseteq (V_{\vec{\Gamma}} \times V_\Gamma) \backslash \Delta$
    - $\Delta = \{(x, x) \mid x \in V_{\vec{\Gamma}}\}$
- $(x, y) \in E_{\vec{\Gamma}}$, then it said x dominates y.

POWER GRAPH

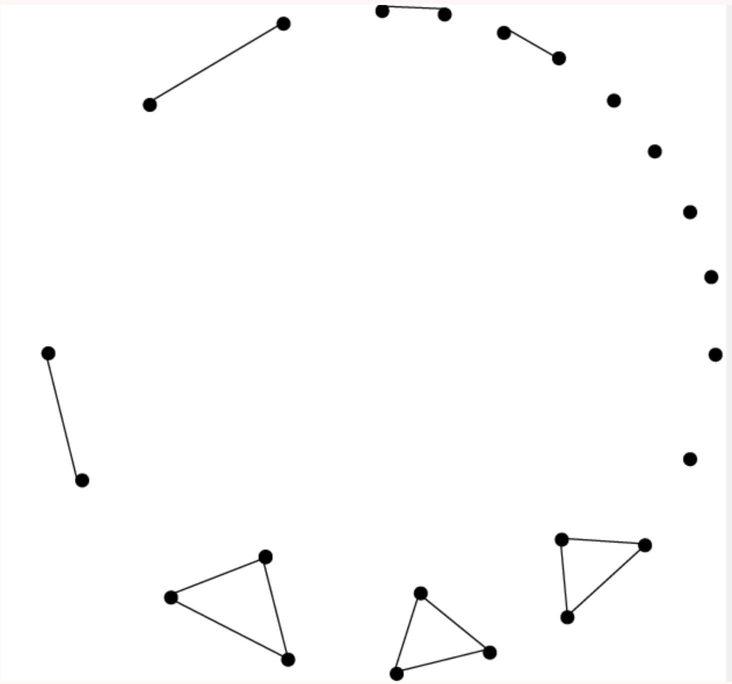
- Power Graph:- G be a group
  - Denoted as P(G)
  - $V_{P(G)}$ is the vertex set
  - $E_{P(G)}$ is the edge set
  - $e = \{x, y\} \in E_{P(G)}$
    - There exist m
    - $X = y^m$
    - $Y = x^m$
- S_4 is an example of cyclic group
- recognize only the star vertex as the identity element
- Observasion:- for all $x \in V_{P(G)}$, we have that $\deg(x) \geq o(x) - 1$
  - $\{x, y\} \in E_{P(G)}$ for all $y \in \langle x \rangle \setminus \{x\}$
  - we have $|\langle x \rangle| = o(x)$

EXAMPLE



PROPER POWER GRAPH

- Proper Power Graph:
  - Denoted by $P^*(G)$
  - $P^*(G)$ is an induced subgraph by G\{1}



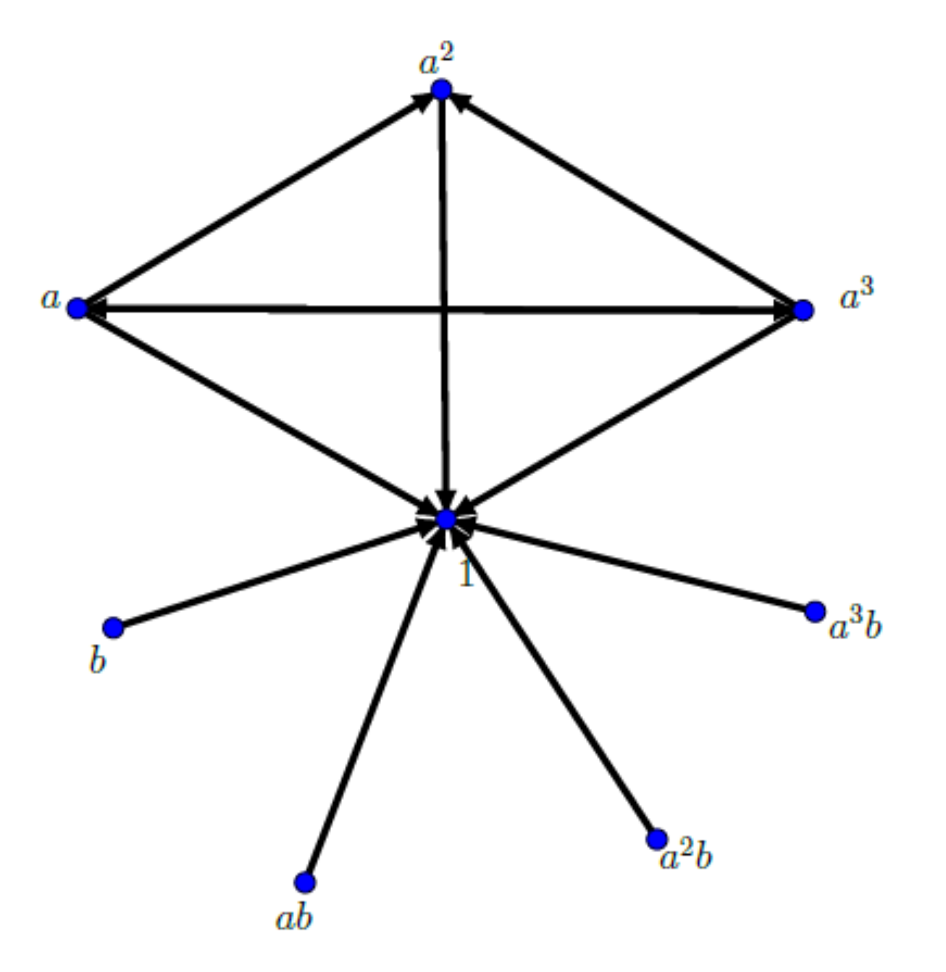
PROPERTIES:-

- Lemma 1:- Let G be a group and H G. Then $P(G)_H = P(H)$.
    - $V_{P(H)} = H = V_{P(G)H}$
    - $e = \{x, y\} \in E_{P(H)}$ iff $e \in E_{P(G)H}$
    - $\exists$ m such that $x = y^m$ or $y = x^m$.
- Lemma 2:- L Let G be a cyclic group. If G is not a p-group, then for all $x \in G \setminus \{1\}$ with $\langle x \rangle = G$ there exists $y \in G$ such that x and y are not joined in P(G).
- Corollary :-
    - Let G be a cyclic group. Then P(G) is a complete graph if and only if G is cyclic of prime power order.

DIRECTED POWER GRAPH

- Defnition:- Let G be a group. The directed power graph of G,
  - denoted $\vec{p}(\mathrm{G})$
  - V $_{\vec{p}(\mathrm{G})} = G$
  - (x, y) ∈ A $_{\vec{p}(\mathrm{G})}$ I
    - f x ≠ y
    - ∃ a positive integer m
      - $y = x^m$.
- Example:- dihedral group of 8 elements: D4 = {1, a, a^2, a^3, b, ab, a^2b, a^3b}
- Lemma 1:- Let G be a group. Then $\vec{p}(\mathrm{G})$ is transitive.
  - Let assume x, y, z ∈ G
  - (x, y), (y, z) ∈ A
  - ∃ m s.t. $y = x^m$
  - ∃ n s.t. $z = y^n$
  - $Z = x^{mn}$

EXAMPLE



SUMMARY

- We studied Group and Graph.
- We studied power graph.
- We studied Proper Power graph.
- We studied Directed Power Graph
- Open Question:-
    - Do there exists a formula for the maximum length of a cycle/path of a power graph?

THANK YOU

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