How many distinct graphs are there on four vertices

Project Report

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April 21, 2024

1 Introduction

This elementary problem in graph theory will be so much calculative if we try to solve only using graph theoretical tools. Even in graphs with 4 vertices we have to look at $2^6 = 64$ graphs and find relations between them . But with the help of group theoretical tool group action this calculation will become very simple.

2 Recollection of basic definitions and results

2.1 Group Operations

Definition 2.1. *(Group Action) An operation of a group G on a set S is a rule for combining an element g of G and an element s of S to get another element of S. In other words, it is a map $G \times S \to S$. An operation is required to satisfy the following axioms:*

1. *$1 \star s = s$ for all $s \in S$.(Here 1 is the identity of G.)*
2. *associative law: $(gg') \star s = g \star (g' \star s)$, for all $g, g' \in G$ and all $s \in S$.*

Definition 2.2. *(Orbit) Given an operation of a group G on a set S, an element s will be sent to various other elements by the group operation. We collect together those elements, obtaining a subset called the orbit O_s of s:*

$$O_s = \{s' \in S : s' = g \star s \text{ for some } g \text{ in } G\}.$$

The orbits for a group action are equivalence classes for the equivalence relation

$$s \sim s' \text{ if } s' = gs, \text{ for some } g \text{ in } G. \text{ Hence, the orbits partitions the set } S.$$

Note that if G is finite, then the number of elements in the orbit of s is at most equal to the order of G, but it might be less. We note the following properties of orbits.

1. For every $s \in S$ we have $s \in O_s$ since $s = 1 \star s$.
2. If $t \in O_s$ for some $s \in S$ then $O_t = O_s$. To see this note that $t = g \star s$ (for some $g \in G$) means that $h \star t = h \star (g \star s) = hg \star s$ for all $h \in G$. Thus every element of O_t belongs to O_s. On the other hand $t = g \star s$ implies that $s = g^{-1} \star t$, so $s \in O_t$and hence $O_s \subseteq O_t$.
3. It follows from above that if some element r of S belongs to the orbits of both s and t, then O_r is equal to both O_s and O_t, so these are equal to each other. So if the orbits determined by two different elements intersect, then they coincide fully. The alternative is that they don't intersect at all.

4. So the action of G partitions the set S into a collection of disjoint subsets, which are orbits. Note that G acts separately upon each orbit, in the sense that elements of G do not move elements of S from one orbit to another, they only move elements around within their own orbit.

5. An action with only one orbit is called transitive. If G acts transitively on the set S, it means that given any elements s, t of S there is an element g of G for which $g \star s = t$.

Definition 2.3. *(Stabilizer) The stabilizer of an element s of S is the set of group elements that leave s fixed. It is a subgroup of G that we often denote by G_s:*

$$G_s = \{g \in G : gs = s\}.$$

Example 2.4 Let n be a positive integer. The group $G = S_n$ acts on the set $S = \{1, 2, \cdots, n\}$ by $\sigma \star i = \sigma(i)$ for all $i \in \{1, 2, \cdots, n\}$.

Lemma 2.5. *Let G be a group acting on a set S and let $s \in S$. Then G_s is a subgroup of G.*

Proof The the identity element of G belongs to G_s is immediate from our definition of group action. Suppose now that $g, h \in G_s$. Then $gh \star s = g \star (h \star s) = g \star s = s$, so $gh \in G_s$ and G_s is closed under the group operation of G. Finally, if $g \in G_s$ then $g \star s = s$ and so $g^{-1} \star (g \star s) = g^{-1} \star s$. Also $g^{-1} \star (g \star s) = (g^{-1}g) \star s = id \star s = s$. Hence $g^{-1} \star s = s$ which means $g^{-1} \in G_s$. Hence G_s is a subgroup of G. □

Proposition 2.6. *Let S be a set on which a group G operates, and let s be an element of S. Let H and O_s be the stabilizer and orbit of s, respectively. There is a bijective map $\Phi : G/H \to O_s$ defined by $[aH] \mapsto as$. This map is compatible with the operations of the group: $\Phi(g[C]) = g\Phi([C])$ for every coset C and every element g in G.*

Proof It is clear that the map Φ defined in the statement of the proposition will be compatible with the operation of the group, if it exists. Symbolically, Φ simply replaces H by the symbol s. What is not so clear is that the rule $[gH] \mapsto gs$ defines a map at all. Since many symbols gH represent the same coset, we must show that if a and b are group elements, and if the cosets aH and bH are equal, then as and bs are equal too. Suppose that $aH = bH$. Then $a^{-1}b$ is in H. Since H is the stabilizer of s, $a^{-1}bs = s$, and therefore $as = bs$. Our definition is legitimate, and reading this reasoning backward, we also see that Φ is an injective map. Since Φ carries $[gH]$ to gs, which can be arbitrary element of O_s, Φ is surjective as well as injective. □

2.2 The Counting Formula

Let H be a subgroup of a finite group G. As we know, all cosets of H in G have the same number of elements, and with the notation G/H for the set of cosets, the order $|G/H|$ is what is called the index $[G:H]$ of H in G. The Counting Formula:

$$|G| = |H||G/H|. \tag{1}$$

There is a similar formula for an orbit of any group operation:

Theorem 2.7. *(Orbit-Stabilizer Theorem) Let S be a finite set on which a finite group G operates, and let O_s and G_s be the orbit and stabilizer of an element s of S. Then*

$$|G| = |G_s||O_s|, \text{ or}$$
$$\text{(order of } G\text{)=(order of stabilizer)} \cdot \text{(order of orbit).}$$

Proof This follows from counting formula (1) and proposition (2.6).

2.3 Burnside's Theorem

Burnside's Theorem will allow us to count the orbits. This is the corollary of Orbit-Stabilizer Theorem.

Definition 2.8. *If group G acts on S and $g \in G$, $fix(g)$ is the set of elements of S, which are fixed by g.*

$$fix(g) = \{s \in S : g \star s = s\}.$$

Theorem 2.9. *(Burnside's Theorem) If a finite group G acts on a finite S then the number of orbits that make up S is:*

$$|O| = \tfrac{1}{|G|} \sum_{g \in G} |fix(g)|.$$

Proof Let O be the set of orbits. We number the orbits that make up S arbitrarily, as $O_1, \cdots, O_k$. Consider the sum

$$\begin{aligned}
\sum_{s \in S} |G_s| &= \sum_{i=1}^{k} \sum_{s \in O_i}^{k} |G_s| \\
&= \sum_{i=1}^{k} |O_i||G_s| \\
&= \sum_{i=1}^{k} |O_i| \frac{|G|}{|O_s|} \\
&= \sum_{i=1}^{k} |G| \\
&= |G| \sum_{i=1}^{k} 1 \\
&= |G||O|
\end{aligned}$$

Then $|O| = \frac{1}{|G|} \sum_{s \in S} |G_s|$. Now,

$$\begin{aligned}\sum_{s \in S} |G_s| &= \sum_{s \in S} \sum_{g \in G_s} 1 \\ &= \sum_{g \in G} \sum_{s \in fix(g)} 1 \\ &= \sum_{g \in G} |fix(g)|.\end{aligned}$$

Therefore $|O| = \frac{1}{|G|} \sum_{g \in G} |fix(g)|$.

2.4 What is Graph

Definition 2.10. *(Graph) A graph $\mathcal{G}$ is a triple consisting of a vertex set V, an edge set E, and a relation that associates with each edge two vertices (not necessarily distinct) called its endpoints. We draw a graph on paper by placing each vertex at a point and representing each edge by a line joining the locations of its endpoints.*

Example 2.11 In the graph in figure (1), the vertex set is $\{x, y, z, w\}$, the edge set is $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ and the assignment of endpoints to edges can be read from the picture.

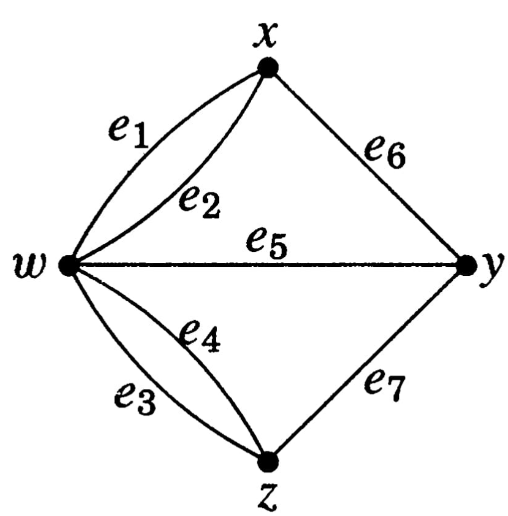

Figure 1:

Definition 2.12. *(Simple graph) A simple graph is a graph having no loops or multiple edges. We specify a simple graph by its vertex set and edge set, treating the edge set as a set of unordered pairs of vertices and writing $e = \{u, v\}$ for an edge e with endpoints u and v.*

Example 2.13 On the below are two drawings of a simple graph. The vertex set is $\{u, v, w, x, y\}$ and the edge set is $\{\{u, v\}, \{u, w\}, \{u, x\}, \{v, x\}, \{v, w\}, \{x, w\}, \{x, y\}\}$.

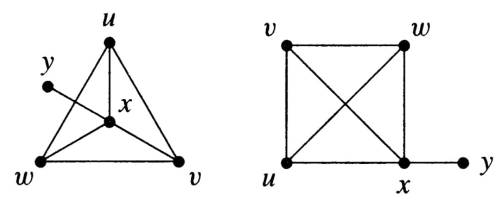

Figure 2:

Definition 2.14. *(Graph isomorphism) An isomorphism from a simple graph $\mathcal{G}$ to a simple graph $\mathcal{H}$ is a bijection $f : V(\mathcal{G}) \to V(\mathcal{H})$ such that $\{u, v\} \in E(\mathcal{G})$ if and only if $\{f(u), f(v)\} \in E(\mathcal{H})$. We say "$\mathcal{G}$ is isomorphic to $\mathcal{H}$".*

Example 2.15 *The graphs $\mathcal{G}$ and $\mathcal{H}$ drawn below are graph with four vertices and four edges. Define the function $f : V(\mathcal{G}) \to V(\mathcal{H})$ by $f(w) = a,\ f(x) = d,\ f(y) = b,\ f(z) = c$. f is a bijection and check that f preserves edges and non-edges. So, f is an isomorphism.*

Figure 3: graph $\mathcal{G}$ and graph $\mathcal{H}$

3 How many different graphs are there on four vertices

In this section we will solve the problem stated in the title. How to proceed:

(a) We will consider a group G and a set S and will define a group operation.

(b) Now our graph theory problem has been converted to group theory problem. The way we should define our operation so that One fix orbit will contain all the isomorphic graphs.

(c) Now our work is to count the number of orbits. Burnside's Theorem will do that.

Problem 1. *How many different graphs are there on four vertices? In this case "graphs" are "simple graph" and "different" means "non-isomorphic".*

Solution 1. *Consider the group G as permutations of the four vertices, so $G = S_4$ and S is the set of all graphs on four vertices. Operation of an element is permuting the vertices with relabeling the edges.*
The complete graph K_4 with four vertices have six edges. Here we are labelling vertex set as $\{1, 2, 3, 4\}$ and labelling edge set as $\{e_1 = \{1, 2\}, e_2 = \{1, 3\}, e_3 = \{1, 4\}, e_4 = \{2, 3\}, e_5 = \{2, 4\}, e_6 = \{3, 4\}\}$. If a particular edge is present in a graph we put 'P', otherwise we will put 'A'. Now we will find cardinality of fix(g), for all $g \in G$.

(a) Identity permutation : It will fix all the graphs. So, $|fix(1)| = 2^6$.

(b) One $2-$cycle : Consider $g = (12)$ and the graph s

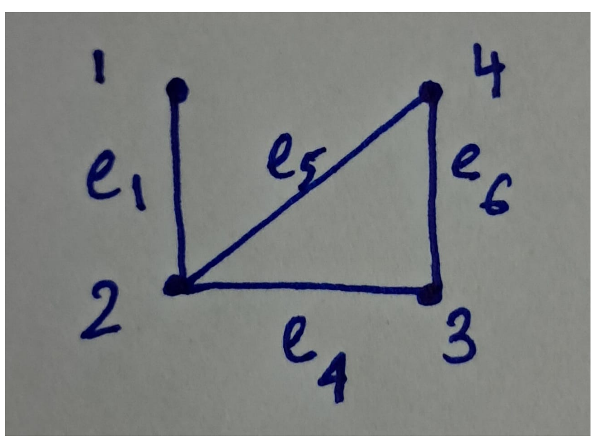

Figure 4: graph s

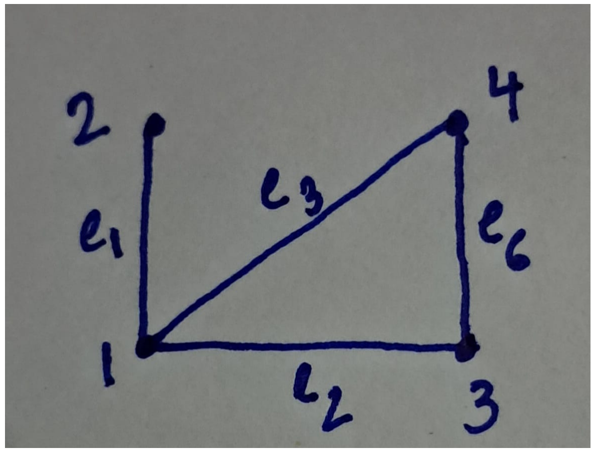

Figure 5: graph $(12) \star s$

We can use (P, A, A, P, P, P) to represent the graph s. After operation representation of the graph is (P, P, P, A, A, P). Two graphs are identical if their representations are same.
Now think of any graph on which cycle (12) *is acting. If* e_1 *is present in s, then after operation it should present. Depending on* e_2*, the edge* e_4 *will be present or absent, depending on* e_3*,* e_5 *will be present or absent and if* e_6 *is present in s then* e_6 *should present in the graph* $(12) \star s$*. Hence the contribution is* $2 \times 2 \times 2 \times 2 = 2^4$*.* S_4 *has 6 2−cycle, so the contribution of all is* 6×2^4.

(c) *Two* $2-cycle$: *Consider* $g = (12)(34)$*.* e_1 *will remain as it is,* e_2 *will label as* e_5 *and vice-versa,* e_3 *will label as* e_4 *and vice-versa,* e_6 *will remain as it is. So, one cycle contribute* $2 \times 2 \times 2 \times 2 = 2^4$*. And* S_4 *have* $\frac{1}{2}{}^4C_2 = 3$ *many permutation, so the contribution of all is* 3×2^4.

(d) $3-cycle$: *Consider* $g = (123)$*. After operation* $e_1 = \{1, 2\}$ *will label as* $e_4 = \{2, 3\}$ *or* $e_2 = \{1, 3\}$ *and vice-versa,* $e_3 = \{1, 4\}$ *will label as* $e_5 = \{2, 4\}$ *or* $e_6 = \{3, 4\}$ *and vice-versa. There are* $4 \times 2 = 8$ *such permutations of vertives, so the contribution of all is* $8 \times (2 \times 2) = 8 \times 2^2$.

(e) $4-cycle$: *Consider* $g = (1234)$*. After operation* $e_1 = \{1, 2\}$ *will label as* $e_4 = \{2, 3\}$ *or* $e_6\{3, 4\}$ *or* $e_3 = \{1, 4\}$ *and vice-versa;* $e_2 = \{1, 3\}$ *will label as* $e_5 = \{2, 4\}$ *and vice-versa. There are* $3! = 6$ *such permutations of vertices, so the contribution of all is* $6 \times (2 \times 2) = 6 \times 2^2$.

(f) $4-cycle$: *Consider* $g = (1234)$*. After operation* $e_1 = \{1, 2\}$ *will label as* $e_4 = \{2, 3\}$ *or* $e_6\{3, 4\}$ *or* $e_3 = \{1, 4\}$ *and vice-versa;* $e_2 = \{1, 3\}$ *will label as* $e_5 = \{2, 4\}$ *and vice-versa. There are* $3! = 6$ *such permutations of vertices, so the contribution of all is* $6 \times (2 \times 2) = 6 \times 2^2$.

Therefore, number of orbits $= \frac{1}{4!}(2^6 + 6 \cdot 2^4 + 3 \cdot 2^4 + 8 \cdot 2^2 + 6 \cdot 2^2) = 11$*. Thus, the number of distinct graph on four vertices is* 11.

$\square$

It is possible, though a bit difficult, to see that for n vertices the result is:

$$f(n) = \sum_j \prod_{k=1}^{n} \frac{1}{k^{j_k} j_k!} \prod_{k=1}^{\lfloor n/2 \rfloor} 2^{k j_k} \prod_{k=1}^{\lfloor (n-1)/2 \rfloor} 2^{k j_{2k+1}} \prod_{k=1}^{\lfloor n/2 \rfloor} \binom{j_k}{2} \prod_{1 \leq r < s \leq n-1} \gcd(r,s)^{j_r j_s}$$

where the sum is over all partitions $j = (j_1, j_2, \cdots, j_n)$ of n, that is, over all j such that $j_1 + 2j_2 + 3j_3 + \cdots + nj_n = n$, and $C(m,2) = {}^4C_2$. With this formula and a computer it is easy to compute the number of graphs when n is not too large; for example, $f(5) = 34$, so there are 34 different five-vertex graphs.

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