How many different graphs are there on four vertices

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Definition 1: Group Action

An operation of a group G on a set S is a rule for combining an element g of G and an element s of S to get another element of S. In other words, it is a map $G \times S \to S$. An operation is required to satisfy the following axioms:

1. $1 \star s = s$ for all $s \in S$.(Here 1 is the identity of G.)
2. associative law: $(gg') \star s = g \star (g' \star s)$, for all $g, g' \in G$ and all $s \in S$.

Definition 2: Orbit O_s of s

$\mathrm{O}_s = \{s' \in S : s' = g \star s$ for some g in G $\}$

The orbits for a group action are equivalence classes for the equivalence relation $s \sim s'$ if $s' = g \star s$, for some $g \in G$. The orbits partitions the set S.

Definition 3: Stabilizer of an element

The *stabilizer* of an element s of S is the set of group elements that leave s fixed. It is a subgroup of G that we denote by G_s: $G_s = \{g \in G : g \star s = s\}$.

Example: Let n be a positive integer. The group $G = S_n$ acts on the set $S = \{1, 2, \cdots, n\}$ by $\sigma \star i = \sigma(i)$ for all $i \in \{1, 2, \cdots, n\}$.

Theorem 1: Orbit Stabilizer Theorem

Let S be a finite set on which a finite group G operates, and let O_s and G_s be the orbit and stabilizer of an element s of S. Then

$$|G| = |G_s||O_s|, \text{ or}$$
$$\text{(order of } G\text{)=(order of stabilizer)} \cdot \text{(order of orbit).}$$

Burnside's Theorem will allow us to count the orbits. This is the corollary of Orbit-Stabilizer Theorem.

Definition 4

If group G acts on S and $g \in G$, fix(g) is the set of elements of S, which are fixed by g.
fix$(g) = \{s \in S : g \star s = s\}$

Theorem 2: Burnside's Theorem

If a finite group G acts on a finite S then the number of orbits that make up S is:

$$|O| = \frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|.$$

Problem: How many different graphs are there on four vertices? In this case "different" means "non-isomorphic".

Solution: Consider the group G is permutations of the four vertices, so $G = S_4$ and S is the set of all graphs on four vertices. How G is acting on S: it is renaming the edges of a graph.

The complete graph K_4 with four vertices have six edges. Labelling edges as $\{e_1 = \{1,2\}, e_2 = \{1,3\}, e_3 = \{1,4\}, e_4 = \{2,3\}, e_5 = \{2,4\}, e_6 = \{3,4\}\}$. If a particular edge is present in a graph we put 'P', otherwise we will put 'A'. Now we will find cardinality of fix(g), for all $g \in G$.

$\textit{Identity permutation}$: It will fix all the graphs. So, $|\text{fix}(1)| = 2^6$.

$one\ 2-cycle$: Consider $g=(12)$ and the graph s

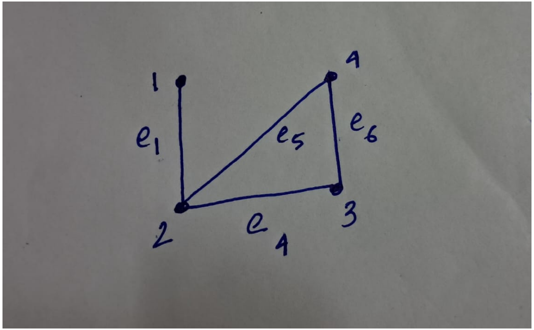

Figure: The graph s

We can use (P, A, A, P, P, P) to represent the graph s.

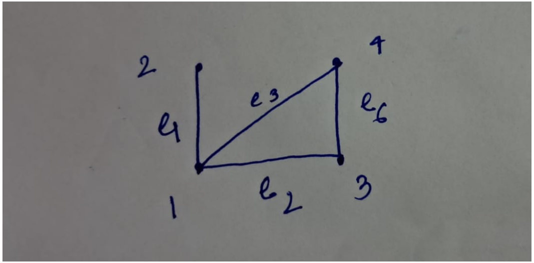

Figure: The graph $(12) \star s$

Here representation of the graph is (P, P, P, A, A, P).
Now think of any graph on which (12) is acting. If e_1 is present in s, then after operation it should present. Depending on e_2, the edge e_4 will be present or absent, depending on e_3, e_5 will be present or absent and if e_6 is present in s then e_6 should present in the graph $(12) \star s$. Hence the contribution is $2 \times 2 \times 2 \times 2 = 2^4$. S_4 has 6 $2-$cycle, so the contribution of all is 6×2^4.

$Two\ 2-cycle$: Consider $g=(12)(34)$. e_1 will remain as it is, e_2 will label as e_5 and vice-versa, $e_3$label as e_4 and vice-versa, e_6 will remain as it is. So, one cycle contribute $2\times 2\times 2\times 2=2^4$. And S_4 have $\frac{1}{2}{}^4C_2=3$ many permutation, so the contribution of all is 3×2^4.

$3 - cycle:$ Consider $g = (123)$. After operation $e_1 = \{1, 2\}$ will label as $e_4 = \{2, 3\}$ or $e_2 = \{1, 3\}$ and vice-versa, $e_3 = \{1, 4\}$ will label as $e_5 = \{2, 4\}$ or $e_6 = \{3, 4\}$ and vice-versa. There are $4 \times 2 = 8$ such permutations of vertives, so the contribution of all is $8 \times (2 \times 2) = 8 \times 2^2$.

$4-cycle:$ Consider $g=(1234)$. After operation $e_1=\{1,2\}$ will label as $e_4=\{2,3\}$ or $e_6\{3,4\}$ or $e_3=\{1,4\}$ and vice-versa; $e_2=\{1,3\}$ will label as $e_5=\{2,4\}$ and vice-versa. There are $3!=6$ such permutations of vertices, so the contribution of all is $6\times(2\times 2)=6\times 2^2$.

Therefore, number of orbits $=\frac{1}{4!}(2^6+6\cdot 2^4+3\cdot 2^4+8\cdot 2^2+6\cdot 2^2)=11$. Thus, the number of distinct graph on four vertices is 11.

Thank You!