Exploring Graph Isomorphism, Equivalence Classes and Automorphisms through Abstract Algebra

Shramana Dey
-JRF in Computer Science

Contents

Introduction

1. **Graph Isomorphism: Beyond Visual Similarity:** Graph isomorphism transcends visual similarity, representing a rigorous mathematical equivalence.
2. **Cross-Domain Significance:** Graph isomorphism plays a pivotal role in network analysis, computer science, and chemistry, highlighting its interdisciplinary importance.
3. **Algebraic Analysis of Graphs:** Algebraic structures, like adjacency and incidence matrices, enable the manipulation and analysis of graphs using matrix algebra.
4. **Equivalence Classes and Graph Isomorphism:** Graph isomorphism forms equivalence relations, leading to the emergence of equivalence classes that partition graphs based on structural similarity.
5. **Group Theory and Automorphism Group:** Group theory aids in studying graph symmetries, with the automorphism group revealing insights into the symmetrical properties of a graph.

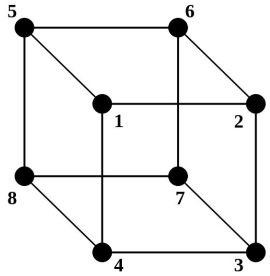
Graph Isomorphism

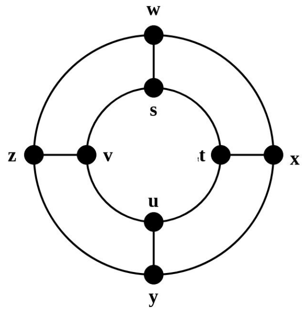
Given a graph G(V, E) and G′(V ′,E′), G and G′ are said to be isomorphic if they satisfy the following necessary condition:

- ❑ They must have the same number of vertices.
- ❑ They must have the same number of edges.
- ❑ They must have equal number of vertices with a given degree.

There exists a bijection map between the vertex set of the graph G and the vertex set of graph G’, defined as $f : V_G \rightarrow V_{G'}$.

Example of one such f is: f (1) = s, f (2) = t, f (3) = u, f (4) = v, f (5) = w, f (6) = x, f (7) = y, f (8) = z Note that, f has the property that neighbors N(v) of a vertex v in graph G is mapped to the neighbors N(f (v)) where, f (v) is the image of v in G′.





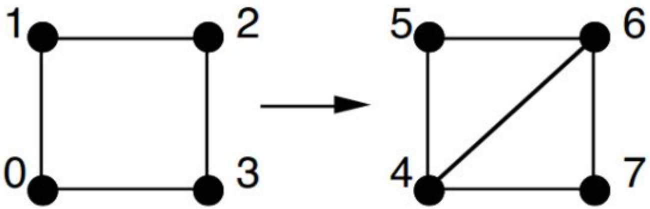
Graph Isomorphism

Let G and H be two simple graphs. A function $f : V_G \to V_H$ preserves **adjacency** if for every pair of adjacent vertices u and v in graph G, the vertices $f(u)$ and $f(v)$ are adjacent in graph H. Similarly, f preserves **non-adjacency** if $f(u)$ and $f(v)$ are non-adjacent whenever u and v are non-adjacent.
A bijection $f : V_G \to V_H$ between two simple graphs G and H is structure-preserving if it preserves both adjacency and non-adjacency. That is, for every pair of vertices u, v in G, u and v are adjacent in $G \iff f(u)$ and $f(v)$ are adjacent in H.

Two simple graphs G and H are isomorphic, denoted $G \cong H$, if $\exists$ a structure-preserving bijection $f: V_G \to V_H$. Such a function f is called an isomorphism from G to H.

Graph Isomorphism

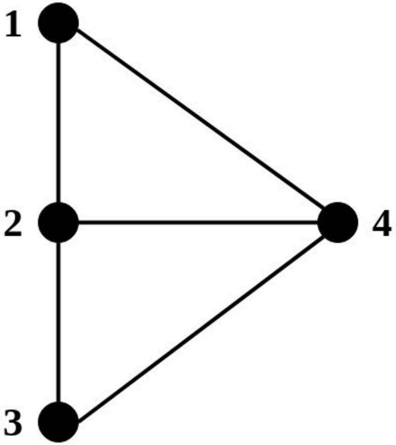
(a) Preserves adjacency and non-adjacency, but not bijective

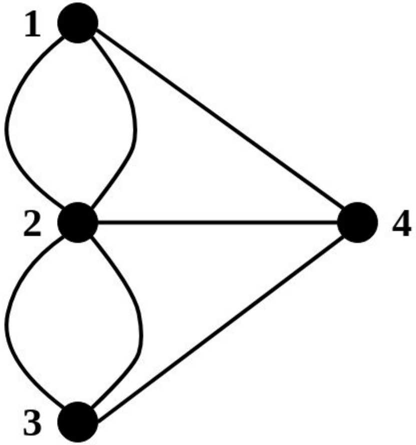


(b) Bijective and adjacency-preserving, but not an isomorphism

The vertex map is defined as (a) $j \to j \bmod 2$, (b) $j \to j + 4$

Graph Isomorphism





Are these two graphs Isomorphic?

To adapt to general graphs, we define:
A bijection $f : V_G \to V_H$ between two graphs G and H, is structure-preserving if (1) the number of edges (even if 0) between every pair of distinct vertices u and v in graph G equals the number of edges between their images $f(u)$ and $f(v)$ in graph H, and (2) the number of self-loops at each vertex x in G equals the number of self-loops at the vertex $f(x)$ in H.

Properties of Graph Isomorphism

Proposition 2.1 *Let G and H be isomorphic graphs. Then they have the same number of vertices and edges.*

Proposition 2.2 *Let $f : G \to H$ be a graph isomorphism and let $v \in V_G$.Then $deg(f(v)) = deg(v)$.*

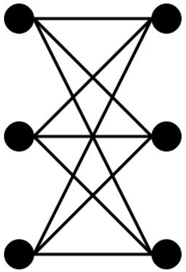
Properties of Graph Isomorphism

Proposition 2.3 *Let G and H be isomorphic graphs. Then they have the same degree sequence.*
OR
If G and H are isomorphic graphs, then the degrees of the vertices of G are the same as the degrees of the vertices of H.

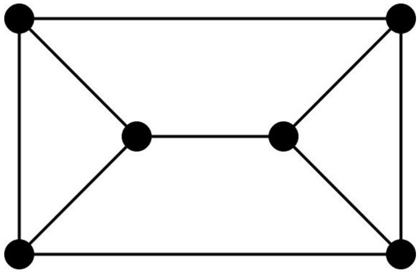
Proposition 2.4 *Let $f : G \to H$ be a graph isomorphism and $e \in E_G$. Then the endpoints of edge $f(e)$ have the same degrees as the endpoints of e.*

Equivalence Classes formed by Isomorphism Graphs

- The relation isomorphism between graphs is an equivalence relation.
- Given by graph X, the set of graphs isomorphic to X is called the isomorphism class of X. The isomorphism classes are the equivalence classes that partition the set of graphs with vertex set V .
- Each equivalence class under isomorphism relation is called an isomorphism type.
- Alongside there are two 3-regular graphs With 6 vertices. Let’s check whether they are isomorphic?



K_{33}



Prism Graph

Equivalence Classes formed by Isomorphism Graphs

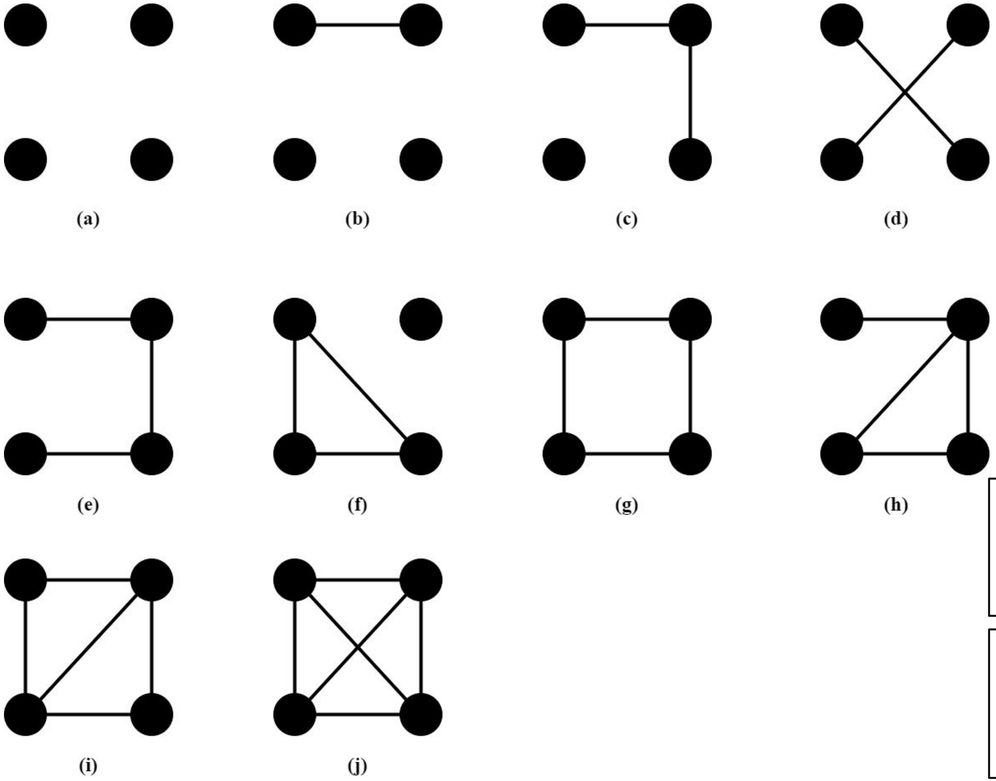


Figure: Equivalence class for 4 vertices

All Graphs of 4 vertices will be isomorphic to any of these graphs.

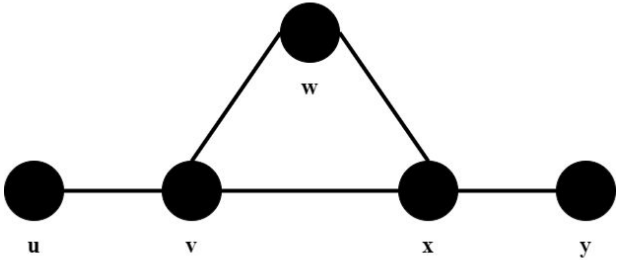
Graph Automorphism

- An isomorphism from a graph G to itself is called an automorphism.
- In the Graph shown alongside, the automorphisms are:
  - The identity automorphism:

    $$e : u \to u, v \to v, w \to w, x \to x, y \to y$$

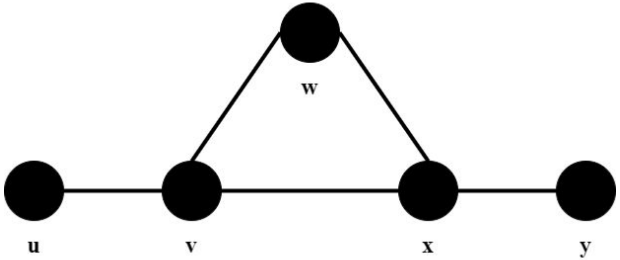
  - The automorphism where the vertices are exchanges:

    $$\alpha : u \to y, v \to x, w \to w, x \to v, y \to u$$



Graph Automorphism

- *Aut(G)* is defined as the **set of all the automorphisms of G**, where G is a graph.
  - *Aut(G)* = {e, α}
- *Aut(G)* forms a Group under Composition.



- An automorphism is therefore a permutation of the vertices and edges of G such that it maps the edges to edges and non-edges to non-edges.
- A graph is said to be **asymmetric** if its automorphism group consists only of the identity map.
- Automorphism describes the symmetry of a graph.

Graph Automorphism

Let's construct the automorphism map for C_4:

Identity map: $e = \begin{pmatrix} u & v & w & x \\ u & v & w & x \end{pmatrix}$

Rotation by 90°: $\alpha_1 = \begin{pmatrix} u & v & w & x \\ v & w & x & u \end{pmatrix}$.

Rotation by 180°: $\alpha_2 = \begin{pmatrix} u & v & w & x \\ w & x & u & v \end{pmatrix}$.

Rotation by 270°: $\alpha_3 = \begin{pmatrix} u & v & w & x \\ x & u & v & w \end{pmatrix}$.

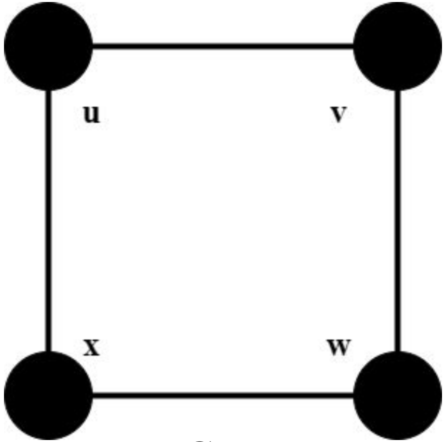
Reflection along the diagonal through vx: $\alpha_4 = \begin{pmatrix} u & v & w & x \\ w & v & x & u \end{pmatrix}$.

Reflection along the diagonal through uw: $\alpha_5 = \begin{pmatrix} u & v & w & x \\ u & x & w & v \end{pmatrix}$.

Reflection along the $x-axis$: $\alpha_6 = \begin{pmatrix} u & v & w & x \\ x & w & v & u \end{pmatrix}$.

Reflection along the $y-axis$: $\alpha_7 = \begin{pmatrix} u & v & w & x \\ v & u & x & w \end{pmatrix}$.

Therefore, $Aut(G) = \{e, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$.

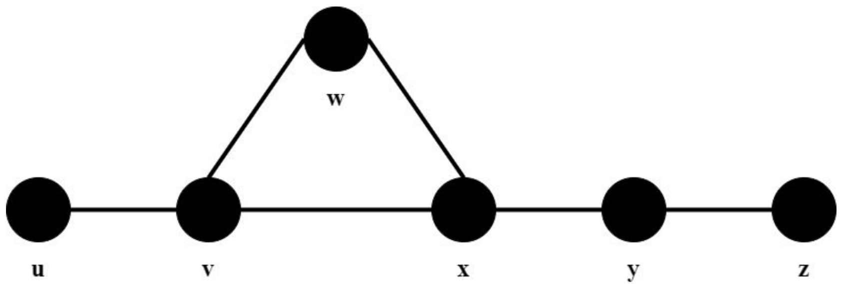


C_4

Thus, all automorphisms of any graph can be described by identity, reflection, or rotation preserving symmetry.

Graph Automorphism

- The more automorphism a graph has, the more symmetric it is.
- The following is an example of smallest automorphic graph.



Properties of Graph Automorphism

Proposition 4.1 *For any graph having n vertices, the complete graph (K_n) among them has the highest number of automorphism.*

Proposition 4.2 *The automorphism group of the complete graph on n vertices with any single edge removed is isomorphic to $S_2 \times S_{n-2}$.*

Idea: Removing an edge from a complete graph leads to a graph having 2 vertices of degree n-2 and n-2 vertices of degree n-1.

Properties of Graph Automorphism

Proposition 4.3 *For the complete bipartite graph,* $K_{n,m}$*, where* $n \geq m$*: If* $n > m$*, then* $Aut(K_{n,m}) \cong S_n \times S_m$.

Idea: Each vertex in one partition has degree either n or m. Exchanges vertices of degree n with another vertex of degree n is only allowed.

Thus, (n!.m!) Permutations are allowed.

Properties of Graph Automorphism

Proposition 4.4 ***Degree Preserving:*** *For all* $u \in V_G$ *and for all* $\phi \in Aut(G)$, $deg(u) = deg(\phi(u))$.

Proposition 4.5 ***Distance Preserving:*** *For all* $u, v \in V_G$ *and for all* $\phi \in Aut(G)$, $d(u, v) = d(\phi(u), \phi(v))$.

Proposition 4.6 *The automorphism group of* G *is equal to the automorphism group of the complement* G.

Properties of Graph Automorphism

Proposition 4.7 *Given a Graph G with n vertices which is not complete, $Aut(G)$ forms a subgroup of $Aut(K_n)$ where K_n is the complete graph with n vertices.*

Idea: G is a subgraph of K_n.
Aut(G) is a group.
$Aut(G) \subseteq Aut(K_n) \cong S_n$

Proposition 4.8 *The order $|Aut(G)|$ of the automorphism group of a graph G of order n is a divisor of $n!$ and equals $n!$ iff $G = K_n$ or $G = K_n'$.*

Idea: We know by Lagrange's theorem, order of a subgroup divides the order of a group.

Frucht's Theorem

Frucht's Theorem Given a finite group τ, there exists a graph G such that $Aut(G)$ is isomorphic to τ.

Some Application of Frucht's Theorem:

1. Provides a connection between finite groups and graphs.
2. Contributes to the development of algorithms for graph isomorphism testing.
3. Has implications in cryptography, particularly in the design of cryptographic protocols and systems based on group theory.
4. Has applications in chemical graph theory, where it can be used to analyze and classify molecular graphs based on their automorphism groups.

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Thank You!