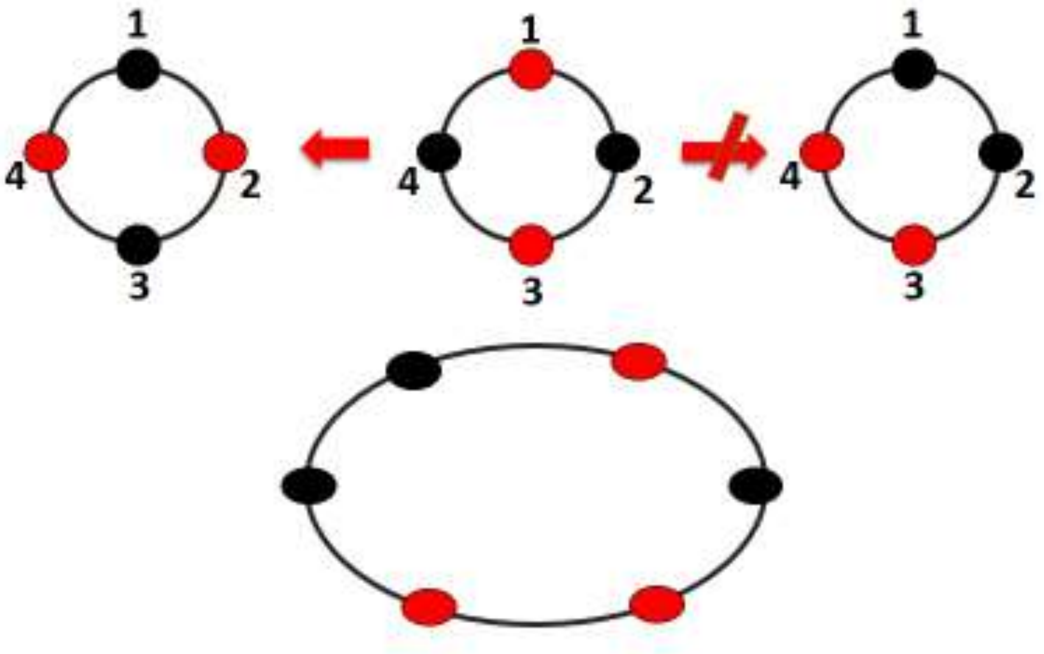
Necklace Design Problem

Introduction

In this article we will discuss about counting number of different designs of a necklace. Suppose a necklace has n beads and beads are of two types. Now different designs can be made by arranging the beads. Again one design can be reached from another design by rotating / reflecting the necklace. We are interested to find minimum number of designs by which all other designs cab be reached. In the figure below, among the 4 bead necklaces, we can reach the left sided necklace from the middle one but cannot reach the right sided necklace.



Algebraic Theorem to solve these kind of problem

Defining Orbit and Stabilizer of a group:

- Consider a group G that acts on a set S.
- Stabilizer of an element of S is defined as follows:

  Let G be a group of permutations of a set S. For each i in S, let $\mathrm{stab}_G(i) = \{\phi \in G \mid \phi(i) = i\}$. We call $\mathrm{stab}_G(i)$ the *stabilizer of i in G.*

- Orbit of an element of S is defined as follows:

  Let G be a group of permutations of a set S. For each s in S, let $\mathrm{orb}_G(s) = \{\phi(s) \mid \phi \in G\}$. The set $\mathrm{orb}_G(s)$ is a subset of S called the *orbit of s under G*. We use $|\mathrm{orb}_G(s)|$ to denote the number of elements in $\mathrm{orb}_G(s)$.

An example of orbit and stabilizer of a group:

- A group G defined as follows on S_8:

$$G = \{(1), (132)(465)(78), (132)(465), (123)(456), (123)(456)(78), (78)\}.$$

Then,

$$\begin{aligned}
&\mathrm{orb}_G(1) = \{1, 3, 2\}, && \mathrm{stab}_G(1) = \{(1), (78)\},\\
&\mathrm{orb}_G(2) = \{2, 1, 3\}, && \mathrm{stab}_G(2) = \{(1), (78)\},\\
&\mathrm{orb}_G(4) = \{4, 6, 5\}, && \mathrm{stab}_G(4) = \{(1), (78)\},\\
&\mathrm{orb}_G(7) = \{7, 8\}, && \mathrm{stab}_G(7) = \{(1), (132)(465), (123)(456)\}.
\end{aligned}$$

Lagrange's Theorem:

Statement:

If G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$. Moreover, the number of distinct left (right) cosets of H in G is $|G|/|H|$.

Proof:

Let $a_1H, a_2H, \ldots, a_rH$ denote the distinct left cosets of H in G. Then, for each a in G, we have $aH = a_iH$ for some i. Also, by property 1 of the lemma, $a \in aH$. Thus, each member of G belongs to one of the cosets a_iH. In symbols,

$$G = a_1H \cup \cdots \cup a_rH.$$

Now, property 4 of the lemma shows that this union is disjoint, so that

$$|G| = |a_1H| + |a_2H| + \cdots + |a_rH|.$$

Finally, since $|a_iH| = |H|$ for each i, we have $|G| = r|H|$.

Orbit Stabilizer Theorem:

Statement:

Let G be a finite group of permutations of a set S. Then, for any i from S, $|G| = |orb_G(i)|\,|stab_G(i)|$.

Proof:

By Lagrange's Theorem, $|G|/|\text{stab}_G(i)|$ is the number of distinct left cosets of $\text{stab}_G(i)$ in G. Thus, it suffices to establish a one-to-one correspondence between the left cosets of $\text{stab}_G(i)$ and the elements in the orbit of i. To do this, we define a correspondence T by mapping the coset $\phi\text{stab}_G(i)$ to $\phi(i)$ under T. To show that T is a well-defined function, we must show that $\alpha\text{stab}_G(i) = \beta\text{stab}_G(i)$ implies $\alpha(i) = \beta(i)$. But $\alpha\text{stab}_G(i) = \beta\text{stab}_G(i)$ implies $\alpha^{-1}\beta \in \text{stab}_G(i)$, so that $(\alpha^{-1}\beta)(i) = i$ and, therefore, $\beta(i) = \alpha(i)$. Reversing the argument from the last step to the first step shows that T is also one-to-one. We conclude the proof by showing that T is onto $\text{orb}_G(i)$. Let $j \in \text{orb}_G(i)$. Then $\alpha(i) = j$ for some $\alpha \in G$ and clearly $T(\alpha\text{stab}_G(i)) = \alpha(i) = j$, so that T is onto.

Burnside Theorem:

Statement:

If G is a finite group of permutations on a set S, then the number of orbits of elements of S under G is

$$\frac{1}{|G|}\sum_{\phi\in G}|\mathrm{fix}(\phi)|.$$

Proof:

Let n denote the number of pairs (ϕ, i), with $\phi \in G$, $i \in S$, and $\phi(i) = i$. We begin by counting these pairs in two ways. First, for each particular ϕ in G, the number of such pairs is exactly $|\mathrm{fix}(\phi)|$. So,

$$n = \sum_{\phi\in G}|\mathrm{fix}(\phi)|. \tag{1}$$

Second, for each particular i in S, observe that $|\mathrm{stab}_G(i)|$ is exactly the number of pairs (ϕ, i) for which $\phi(i) = i$. So,

$$n = \sum_{i\in S}|\mathrm{stab}_G(i)|. \tag{2}$$

It follows from Exercise 33 in Chapter 7 that if s and t are in the same orbit of G, then $\mathrm{orb}_G(s) = \mathrm{orb}_G(t)$, and thus by the Orbit-Stabilizer Theorem (Theorem 7.3) we have $|\mathrm{stab}_G(s)| = |G|/|\mathrm{orb}_G(s)| = |G|/|\mathrm{orb}_G(t)| = |\mathrm{stab}_G(t)|$. So, if we choose $s \in S$ and sum over $\mathrm{orb}_G(s)$, we have

$$\sum_{t\in \mathrm{orb}_G(s)}|\mathrm{stab}_G(t)| = |\mathrm{orb}_G(s)|\,|\mathrm{stab}_G(s)| = |G|. \tag{3}$$

Finally, by summing over all the elements of G, one orbit at a time, it follows from Equations (1), (2), and (3) that

$$\sum_{\phi\in G}|\mathrm{fix}(\phi)| = \sum_{i\in S}|\mathrm{stab}_G(i)| = |G|\cdot(\text{number of orbits})$$

and the result follows.

Solving our problem by this theory:

We have considered two kinds of necklaces. In first kind there are 4 beads and two of them coloured red, other two are black; In second kind 4 beads and all of them can be coloured either red or black. By design fixed by an operation we mean number of designs which remains identical by the operation. Different group operations and number of design fixed by the operations in both cases have been noted in the following table.

Group operation	Number of designs fixed for n=4, k=2	Number of designs fixed for n=4
Identity 0 degree rotation	6	16
90 degree rotation	0	2
180 degree rotation	2	4
270 degree rotation	0	2
Horizontal reflection	2	4
Vertical reflection	2	4
1st diagonal reflection	2	8
2nd diagonal reflection	2	8
Total	16	48
Number of classes	2	6

Number of classes are the number of designs that are needed to reach all possible design.

Calculation details by Burnside Theorem:

$$For\ the\ first\ case,\ \frac{1}{8}(6+0+2+0+2+2+2+2) = \frac{16}{8} = 2$$

$$For\ the\ second\ \ case,\ \frac{1}{8}(16+2+4+2+4+4+8+8) = \frac{48}{8} = 6$$

By application of Burnside theorem we can find the minimum number of different design that is needed to reach all other design in similar kind of problems.