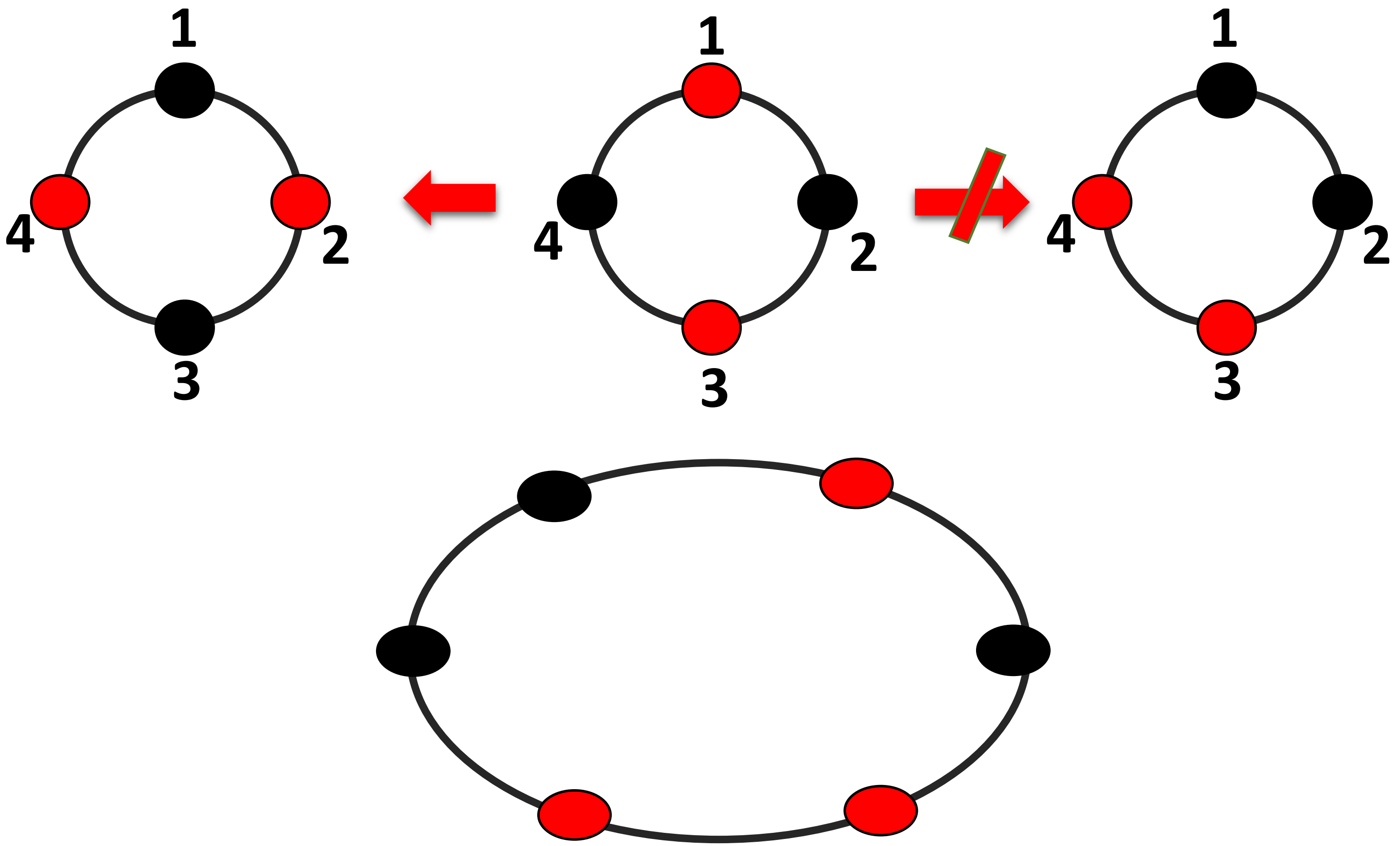
Necklace design problem



Consider design of a necklace with n beads with k of them are coloured red and rests are black. Certainly ${}^{n}C_{k}$ different designs are there. But some design can be achieved by some operations (say rotation/reflection) on other designs. We are interested to find the minimum number of necklaces needed to get all possible designs by operations on those necklaces.

Introducing orbit and stabilizer of a group

- Consider a group G that acts on a set S.
- Stabilizer of an element of S is defined as follows:

Let G be a group of permutations of a set S. For each i in S, let $\text{stab}_G(i) = \{\phi \in G \mid \phi(i) = i\}$. We call $\text{stab}_G(i)$ the *stabilizer of i in G.*

- Orbit of an element of S is defined as follows:

Let G be a group of permutations of a set S. For each s in S, let $\text{orb}_G(s) = \{\phi(s) \mid \phi \in G\}$. The set $\text{orb}_G(s)$ is a subset of S called the *orbit of s under G*. We use $|\text{orb}_G(s)|$ to denote the number of elements in $\text{orb}_G(s)$.

Example of orbit and stabilizer of a group

- A group G defined as follows on S_8:

$$G = \{(1), (132)(465)(78), (132)(465), (123)(456), (123)(456)(78), (78)\}.$$

Then,

$$\mathrm{orb}_G(1) = \{1, 3, 2\}, \qquad \mathrm{stab}_G(1) = \{(1), (78)\},$$
$$\mathrm{orb}_G(2) = \{2, 1, 3\}, \qquad \mathrm{stab}_G(2) = \{(1), (78)\},$$
$$\mathrm{orb}_G(4) = \{4, 6, 5\}, \qquad \mathrm{stab}_G(4) = \{(1), (78)\},$$
$$\mathrm{orb}_G(7) = \{7, 8\}, \qquad \mathrm{stab}_G(7) = \{(1), (132)(465), (123)(456)\}.$$

Burnside Theorem

- Orbit Stabizer theorem states:

Let G be a finite group of permutations of a set S. Then, for any i from S, $|G| = |orb_G(i)|\,|stab_G(i)|$.

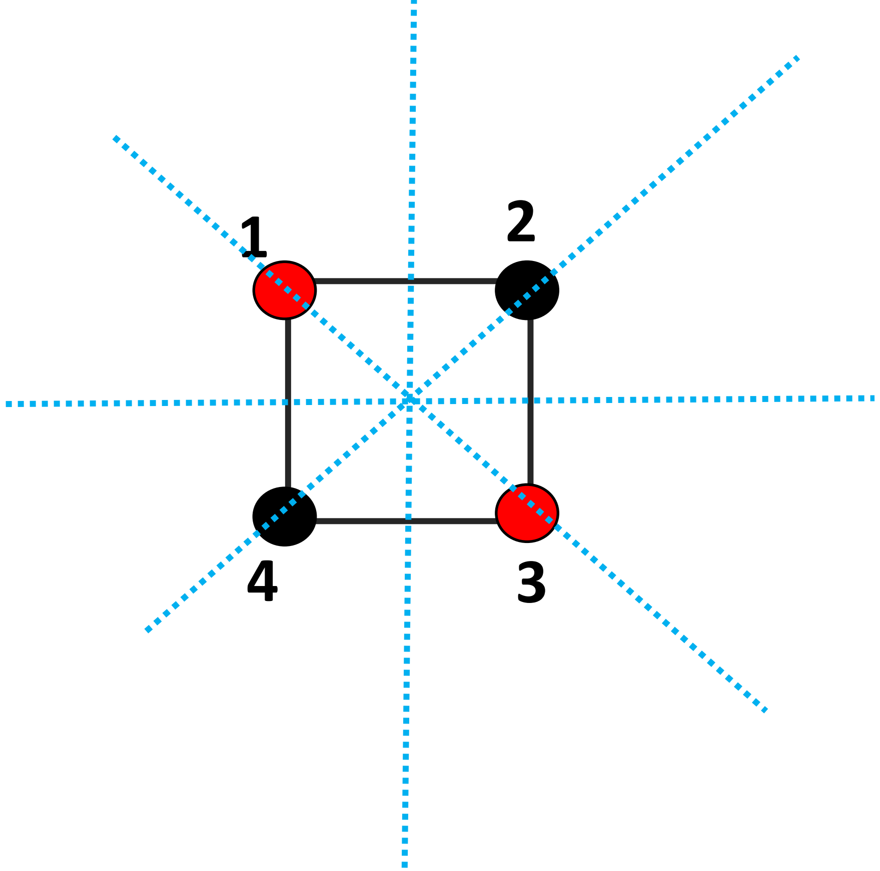
- Defining elements fixed by a group operation ϕ

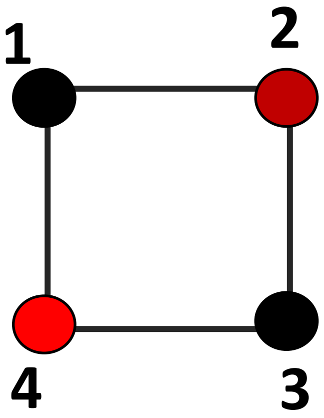
For any group G of permutations on a set S and any ϕ in G, we let $\text{fix}(\phi) = \{i \in S \mid \phi(i) = i\}$. This set is called the *elements fixed by* ϕ (or more simply, "fix of ϕ").

- Burnside Theorem:

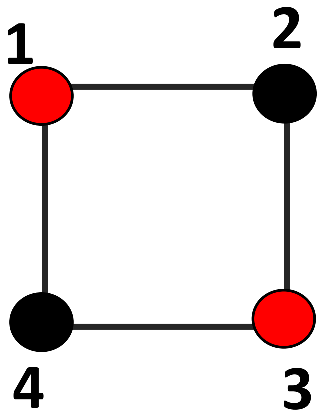
If G is a finite group of permutations on a set S, then the number of orbits of elements of S under G is

$$\frac{1}{|G|} \sum_{\phi \in G} |\text{fix}(\phi)|.$$





Vertical Reflection
90 degree rotation



Diagonal Reflection
180 degree rotation

- So group(G) has operations: 0^0, 90^0, 180^0, 270^0 rotations and vertical, horizontal reflections and reflections by two diagonals.

- The group acts on set S=D_4

Calculating no of classes

Let all the beads of necklace do not have same colour. So at least one consecutive pair of beads exists that have different colours. But for 90 degree rotations all consecutive pairs should have same colour. So 90 degree rotation does not fix any element in this case.
But 90 degree rotation fixes elements when all beads are of same colour.

The no of elements fixed by other group opertaions has been observed.

Group operation	Number of designs fixed for n=4, k=2	Number of designs fixed for n=4
Identity 0 degree rotation	6	16
90 degree rotation	0	2
180 degree rotation	2	4
270 degree rotation	0	2
Horizontal reflection	2	4
Vertical reflection	2	4
1st diagonal reflection	2	8
2nd diagonal reflection	2	8
Total	16	48
Number of classes	2	6

For n=6 and k=3 we have the following table

Type of Element	Number of Elements of This Type	Number of Arrangements Fixed by Type of Element
Identity	1	20
Rotation of order 2 (180°)	1	0
Rotation of order 3 (120° or 240°)	2	2
Rotation of order 6 (60° or 300°)	2	0
Reflection across diagonal	3	4
Reflection across side bisector	3	0

$$From\ Burnside\ Theorem\ \frac{1}{1+1+2+2+3+3}(1.20 + 1.0 + 2.2 + 2.0 + 3.4 + 3.0) = \frac{36}{12} = 3$$

THANK YOU

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