Aggregating Relational Structures

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A toy example

- Please think of a real number x. Let's call it your number.
- We now have a number for each person. How to find an "ICLA number" from these?
- A systematic (aggregation) rule that makes this choice would look like $\sigma : \mathbb{R}^n \to \mathbb{R}$.
- Reals are nice, so we have some well know rules such as mean/median/mode.
- Since this is my presentation, I can also say let's use $\sigma(\bar{x}) = x_{harshit}$.

IS THIS FAIR?

Start of social choice theory

- Consider a finite set of election candidates A, to be chosen by finite voter set $\mathcal{I}$.
- Consider a predicate language $\mathcal{L}$ consisting of a single predicate symbol R of arity $k = 2$.
- Let T be the $\mathcal{L}$ theory that specifies R is anti-symmetric, transitive, irreflexive and trichotomous (full linear order).
- Denote the collection of T models over A by $\mathcal{M}(A)$.
- We want a fair aggregation rule (social welfare function) $\sigma : \mathcal{D} \subseteq \mathcal{M}(A)^{\mathcal{I}} \to \mathcal{M}(A)$.
- Aggregating user preferences over election candidates to find community preference order over candidates (and not just a single winner).

Fairness properties of aggregation

Desirable:

- **(UD)** The voters are not restricted to vote from a small collection of orders.
- **(P)** If all voters prefer *a* over *b* in a voting profile, *a* must be preferred over *b* in community order chosen by σ.
- **(IIA)** If across two voting profiles every voter preserves the relative order for *a* and *b*, the two community orders must also preserve their relative order.

Undesirable:

- **(D)** The aggregation rule simply chooses the preference order of one voter (dictator).

Theorem (Arrow's Impossibility Theorem [Arr12])

For the social choice situation $(A, \mathcal{I}, \mathcal{D}, \sigma)$*, the social welfare function* $\sigma : \mathcal{D} \subseteq \mathcal{M}(A)^{\mathcal{I}} \to \mathcal{M}(A)$ *where* $\mathcal{D}$ *is* UD *satisfying* P *and* IIA *must also satisfy* D*.*

Arrow's Impossibility Theorem

Arrow's Theorem: (Binary)

Antisymmetric

Transitive

Trichotomous

Irreflexive

Results in social choice theory

- **(Negative)**
  - Aggregation of equivalence relations (essentially, aggregating how individuals cluster a set of objects) by Fishburn and Rubinstein [FR86].
  - Aggregation of partial orders on the candidate set by Pini et al [PRVW09].
- **(Positive)**
  - Aggregating single-peaked preferences by Black. [Bla48].
  - Allowing representative dictators by Tangian et al [T^{+}14].

Why study aggregation?

- Understanding properties of mechanisms used to aggregate user preferences in selecting elected representatives, referendums for establishing social justice, and allocating shared resources.
- Make sense of individual data sources. For example, how to decide which roads are most important to improve/maintain when GPS data of citizens' movement available?
- [DLT20] and [PS20] provide good examples of aggregation arising as a key concern in group recommender systems and parsing user review data.

Generalizing to k-ary relations

Consider k-ary relations satisfying the following properties:

- **connected** if, for each pairwise-distinct $\bar{a}_k \in A$, there is a permutation τ of $\{1, ..., k\}$ such that $\bar{a}_k^\tau \in R$.
- **exclusive** if for each pairwise-distinct $\bar{a}_k \in A$, there is a permutation τ of $\{1, ..., k\}$ such that $\bar{a}_k^\tau \notin R$.
- **simplicial transitive** if for each sequence of pairwise-distinct elements $\bar{a}_{k+1}$ for each $j \in \{1, ..., k+1\}$ if $(\bar{a}_{k+1})_k^{+j} \subseteq R$ then $(\bar{a}_{k+1})_k^{-j} \in R$.

Motivating aggregation of higher arity relations

Below are examples of relations satisfying the properties:

- **Seating along a circular table** Consider a party of dinner guests to be seated on circular table in groups of 4 (any cyclic arrangement is fine) and preferences over how every subsets of 4 people should be seated. This quaternary relation R on the party of dinner guests is such that if $(a, b, c, d) \in R$ then all cyclic permutations $(b, c, d, a), (c, d, a, b), (d, a, b, c) \in R$ and no other permutation of $\{a, b, c, d\}$ is in R. This relation also is connected, exclusive and simplicial transitive.
- **Moderate Voters** If each voter always prefers the moderate candidate in any group of 3 candidates, then this voting behaviour can be captured by a "betweenness" relation, with $(a, b, c) \in R_i \Leftrightarrow (c, b, a) \in R_i$ representing the i^{th} voters preference for b over a and c. This relation is connected, exclusive as well as simplicial transitive.

A negative k-ary result

- Consider a predicate language $\mathcal{L}$ consisting of a single predicate symbol R of arity $k \geq 3$.
- Let T be the $\mathcal{L}$ theory that specifies R is connected, exclusive, and simplicial transitive. Denote the collection of T models over A by $\mathcal{M}(A)$.
- We want a fair aggregation rule (social welfare function) $\sigma : \mathcal{D} \subseteq \mathcal{M}(A)^{\mathcal{I}} \rightarrow \mathcal{M}(A)$.

Theorem (B., Kuber)

For the social choice situation $(A, \mathcal{I}, \mathcal{D}, \sigma)$*, the social welfare function* $\sigma : \mathcal{D} \subseteq \mathcal{M}(A)^{\mathcal{I}} \rightarrow \mathcal{M}(A)$ *where* $\mathcal{D}$ *is* UD *satisfying* P *and* IIA *must also satisfy* D*.*

Two specific results: restricted by definitions?

Arrow's Theorem:
(Binary)
Antisymmetric
Transitive
Trichotomous
Irreflexive

(k-ary)
Exclusive
Simplicial Transitive
Connected

Generalizing for binary relations

- Any binary relation R on A can be visualized as a digraph $(A, R \subseteq A^2)$.
- A **contagious** property allows included tuples to affect inclusion of neighboring tuples.
- An **implicative** property forces inclusion of some edges conditional on other edges present.
- A **disjunctive** property forces one of two edges to be included in the relation.

Theorem (Endriss and Grandi [EG17])

For $|A| \geq 3$, any unanimous, grounded, and IIA aggregation rule σ that is collectively rational (preserves) with respect to a digraph property P that is contagious, implicative, and disjunctive must be dictatorial on non-reflexive edges.

Generalizing to *k*-ary?

Arrow's Theorem:
(Binary)
Antisymmetric
Transitive
Trichotomous
Irreflexive

[EG17]:
(Binary)
All Relations
Satisfying
Suitable
Metaproperties

(k-ary)
Exclusive
Simplicial Transitive
Connected

General k-ary Result

- Any k-ary R on A can be visualized as a k-uniform directed hypergraph $(A, R \subseteq A^k)$.
- We define analogous metaproperties that are sufficient to identify k-ary relations where fair aggregation is impossible.

Theorem (B., Kuber)

For $|A| \geq 3$, any unanimous, grounded, and IIA aggregation rule σ that is collectively rational (preserves) with respect to a k-uniform hypergraph property P that is contagious, implicative, and disjunctive must be dictatorial on non-degenerate k-hyperedges.

Final contribution

Arrow's Theorem:
(Binary)
Antisymmetric
Transitive
Trichotomous
Irreflexive

[EG17]:
(Binary)
All Relations
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Existing and future work

- Other generalizations of negative social choice results have been focused on multiwinner elections [KdVV$^+$20] trying to pick a subset of A as opposed to a relation on A.
- How to aggregate individual opinions expressed as simplicial complexes of bounded dimension? Simplicial complexes are useful in applications such as distributed rendering of $3D$-graphics. A group of friends who where every subset forms a group can also represented as a simplicial complex.

Thank you!

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