A note on hybrid modal logic with propositional quantifiers (Work in progress)

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Plan of talk

I **Standard hybrid logic**

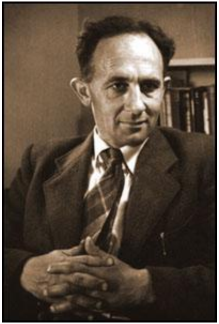
II **Propositional quantifiers and Prior's Q operator**

III **Tableau proof-rules**

Part I

Standard hybrid logic

Hybrid logic was invented by Arthur Prior (1914-1969)



- Prior's aim was to solve a problem in the philosophy of time
- Technically, he increased the expressive power of ordinary modal logic
- First key idea in hybrid logic:
  add nominals to the modal language,
  propositional symbols true at precisely one world/time/person/state/location:
  for example patrick and irina
- Second key idea in hybrid logic:
  build satisfaction statements,
  formulas like $@_{patrick} philosopher$ and $@_{irina} psychologist$

Standard nominals

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> a *It is 12:30 November 22nd 1963*

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Remedy:
Add new propositional symbols a, b, c, . . . called *nominals*

A nominal a is true at exactly one time, so it refers to a time

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$@_a p$ *At 12:30 Nov. 22nd 1963, J.F. Kennedy is shot*

which is about what happens at a particular time

Remedy: For each nominal a add a *satisfaction operator* $@_a$

The satisfaction operator $@_a$ moves the time of evaluation to the time referred to by the nominal a

Thus, a formula $@_a\phi$ is true iff ϕ is true at the time a refers to

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Also, the relation between the few known systems is unclear[2]

These deficiencies can be remedied by hybridization!

For example, hybrid-logical proof-systems can handle different models of time uniformly

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Salvador Dali "The Persistence of Time" 1931

Part II

Propositional quantifiers and Prior's Q operator

The first-order/second-order divide is not innocent!

First-order logic is axiomatisable, but second-order logic is not!

In 1950, Leon Henkin showed how to 'tame' second-order logic:

Instead of interpreting second-order quantifiers as ranging over **all** subsets of the domain of quantification, view them as ranging over **a pre-selected set** of admissible subsets

Hence, instead of working with models dictated by set theory (all the subsets), work with deliberately pre-structured models

The pre-structured models have to satisfy certain intuitive constraints (closure properties)

Definition of nominals using the Q operator

Instead of introducing nominals as a second sort of propositional symbol, Prior sometimes defined them using the Q operator.

> *For 'p is an individual' (or an instant, or a possible total world-state) we write Qp. If we have propositional quantifiers, we can define Qp thus:*

$$Qp = \Diamond p \wedge \forall q(\Box(p \rightarrow q) \vee \Box(p \rightarrow \neg q))$$

Here $\Box$ means *true at all worlds* and $\Diamond$ means *true at some world* (universal modalities)

So Qp says that p is *possible* and *maximal*: p is true *somewhere* and p strictly implies *every* proposition q or its negation (Read $\Box(p \rightarrow q)$ as "p is included in q" etc.)

Interpretation of the $\forall q$ quantifier in the Q operator

If $\forall q$ ranges over **all** subsets of worlds, then Qp says that p is a singleton set, in other words, a standard nominal

But if $\forall q$ is given a general (Henkin) semantics where it ranges over **a pre-selected set** of subsets of worlds[3] then Qp says that p is an atom, that is, a non-empty minimal preselected subset

So the general semantics gives a non-standard 'species' of nominals

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To make the differences between our two species of nominals concrete, it will help to have a proof-system

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Part III

Tableau proof-rules

Tableau proof-rules for the propositional quantifier

We focus on the universal instantiation (UI) rule on the left

$$\begin{array}{c} @_i \forall p \varphi \\ \downarrow \\ @_i \varphi[\psi/p]\dagger \end{array} \qquad \begin{array}{c} \neg @_i \forall p \varphi \\ \downarrow \\ \neg @_i \varphi[q/p]* \end{array}$$

$\dagger$: *where* ψ *is free for p in* φ *and* ψ *does not contain any standard nominal in formula position.*

$*$: *where q is a new propositional symbol.*

The first part of the $\dagger$ side-condition prevents accidental symbol binding (defined in the usual way)

The second part of $\dagger$ is where the distinction between the standard and non-standard nominals becomes important

What is the role of the † restriction on the UI rule?

The unrestricted UI rule is sound wrt. the standard semantics, but this rule is *not* sound wrt. the general semantics

However, the restricted UI rule is sound wrt. the general semantics, and we conjecture that the tableau system is also complete

Thus, if this conjecture is correct, we have found a proof-system that can deal with the new species of nominals!

Summing up:

If the standard semantics is chosen for the propositional quantifiers, the Q operator gives standard nominals

But if the general semantics is chosen, the Q operator gives a new species of nominals

Moreover, a proof-system can be found that is sound and (we think) also complete with respect to the general semantics