Equivalence of pointwise and continuous Semantics of first-order logic with linear constraints

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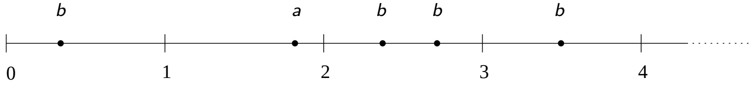
Outline

- Pointwise and continuous interpretations
- First-Order logic with linear constraints
- From continuous to pointwise.
  - Eliminating a single top-level passive quantifier
  - Eliminating all passive quantifiers.
- Future directions.

Timed words

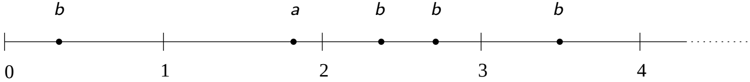
- Timed words [Alur and Dill] are a popular model of real-time behaviours.



- Similar to classical word but each action has a time-stamp.
- Assumption: Time-stamps are progressive.

Quantitative Temporal Logics

- Metric Temporal Logic (MTL) [Koymans 1992, Alur-Feder-Henzinger 1996, Ouaknine-Worrell 2005]
    - aUb "there is a future timepoint at which a b occurs, and till then a occurs."
    - $aU_I b$ "... and the timepoint lies at a distance which lies in the interval I."
    - $\Diamond\varphi \equiv true U\varphi$: "eventually φ."
    - $\Diamond_I\varphi \equiv true U_I\varphi$: "eventually φ at a distance that lies in I."
- Timed Propositional Temporal Logic (TPTL) [Alur-Henzinger 1994].
    - $\Diamond x.(\Diamond y.(a \wedge y = x + 1))$: "There is a future timepoint x and a subsequent timepoint y at which an a occurs and $y = x + 1$."

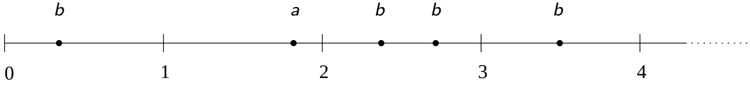


Pointwise vs continuous semantics

Two natural interpretations:

- Pointwise: quantification is over action timepoints in timed word.
- Continuous: quantification is over arbitrary timepoints in timed word.

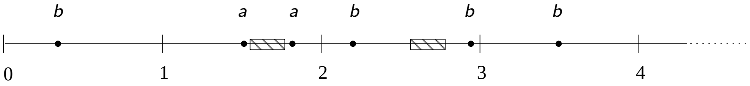
Consider MTL assertion $\Diamond(\Diamond_{[1,1]} a)$ "Eventually there is a timepoint from which we have an action a at distance 1," on timed word below:



False in pointwise semantics but True in continuous semantics.

Typically pointwise less expressive than continuous

- Pointise MTL is less expressive than Continuous MTL.
  - Property "no insertions" can be expressed in continuous MTL but *not* pointwise MTL.



- Also true for other variants of MTL (MTL_S, MTL_{S_I}, MITL).
- What about TPTL?

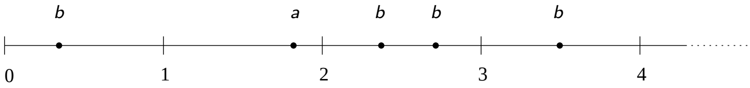
First-Order Logic of linear constraints

- Expressively same as TPTL with "Since" operator.
- Interpreted over timed words.
- $a(x)$: "timepoint x has an a action."
- $x \sim y + c$ where $\sim$ is in $\{<, \leq, =, \geq, >\}$.
- Boolean combinations: $\neg$, $\wedge$, $\vee$.
- First-order quantification: $\exists x \varphi$.

Semantics of $\mathrm{FO}(<, +)$

- Interpreted over timed words.
- $\exists x$ interpreted as
  - "there exists an action point x" (pointwise).
  - "there exists a timepoint x" (continuous).

Example sentence: $\exists x \exists y (a(y) \wedge y = x + 1)$.



Sentence is False in pointwise semantics but True in continuous semantics.

What we show

For a $\mathrm{FO}(<,+)$ sentence φ:

- $L^{pw}(\varphi)$ = set of timed words that satisfy φ in pointwise semantics.
- $L^{c}(\varphi)$ = set of timed words that satisfy φ in continuous semantics.

Theorem

The class of timed languages definable in $\mathrm{FO}(<,+)$ in the pointwise and continuous semantics coincide.

Easy Part: From $\mathrm{FO}^{pw}(<,+)$ to $\mathrm{FO}^{c}(<,+)$

Given φ, find φ' such that $L^{pw}(\varphi) = L^{c}(\varphi')$.
Replace

$$\exists x \psi$$

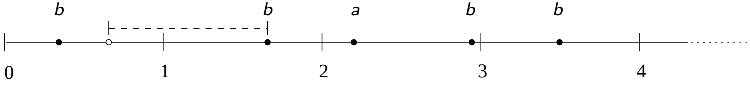
by

$$\exists x \big(\bigvee_{a \in \Sigma} a(x) \ \wedge \ \psi' \big).$$

Difficult Part: From $\mathrm{FO}^c(<,+)$ to $\mathrm{FO}^{pw}(<,+)$

Given $\mathrm{FO}^c(<,+)$ sentence:

$$\exists x(\neg a(x) \wedge 0 \leq x \leq 1 \wedge \exists y(b(y) \wedge y = x + 1))$$



A possible equivalent $\mathrm{FO}^{pw}(<,+)$ formula is:

$$\exists y(b(y) \wedge 1 \leq y \leq 2 \wedge \neg\exists x(a(x) \wedge y = x + 1)).$$

Main idea

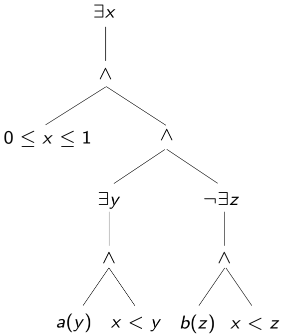
- Go from an $\mathrm{FO}^c(<,+)$ sentence φ to an equivalent actively quantified $\mathrm{FO}^c(<,+)$ sentence φ'.
- Observe that if φ' is actively quantified, then $L^c(\varphi') = L^{pw}(\varphi')$.
- So φ' could be an equivalent $\mathrm{FO}^{pw}(<,+)$ sentence.

Main steps

- First put φ in a normal form.
- Show how to eliminating a single top-level passive quantifier.
- Eliminate all passive quantifiers step by step.

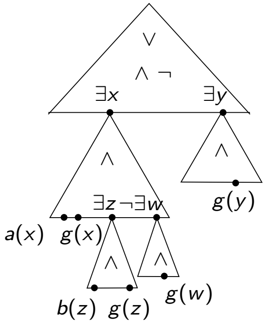
Normal form for $\mathrm{FO}(<,+)$ sentences

- Normal form: Boolean combination of sentences in ∃-normal form.
- Example formula in ∃-normal form:



Procedure to convert to normal form

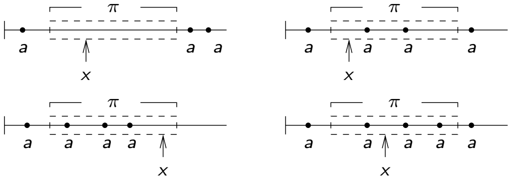
1. Push $\neg$'s downward till $\exists$-nodes or $a(x)$-nodes.
2. Pull $\vee$'s upward (eg. $\exists x(\alpha \vee \beta) \equiv (\exists x \alpha) \vee (\exists y \beta)$).
3. Replace $a(x) \wedge b(x)$ by *false* if $a \neq b$.
4. Replace $\exists x(\neg a(x) \wedge \pi(x) \wedge \alpha)$ by $\psi_1 \vee \psi_2 \vee \psi_3 \vee \psi_4$.



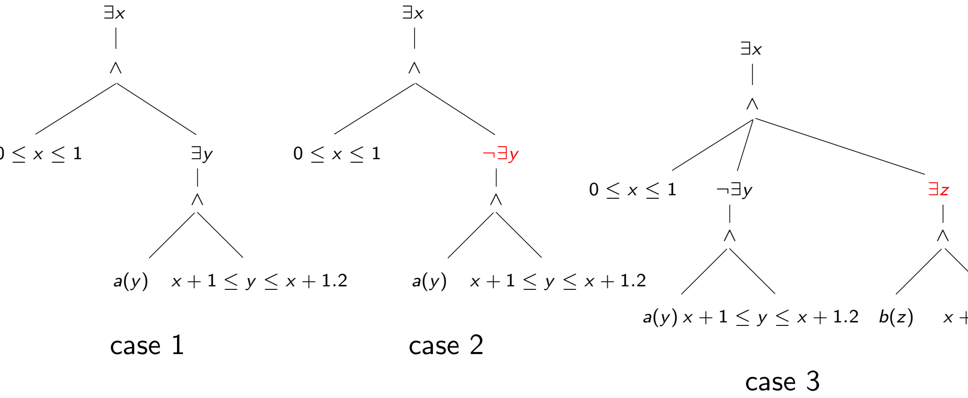
Procedure to convert to normal form

Replace $\exists x(\neg a(x) \wedge \pi(x) \wedge \alpha)$ by $\psi_1 \vee \psi_2 \vee \psi_3 \vee \psi_4$, where:

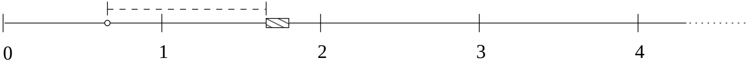
- $\psi_1 = \neg\exists x(a(x) \wedge \pi(x)) \wedge \exists x(\pi(x) \ \wedge \ \alpha)$.
- $\psi_2 = \exists x_l(a(x_l) \wedge \pi[x_l/x] \wedge \neg\exists x'(a(x') \wedge \pi[x'/x] \wedge x' < x_l) \wedge \exists x(\pi(x) \wedge x < x_l \ \wedge \ \alpha))$.
- Similarly ψ_3, ψ_4.

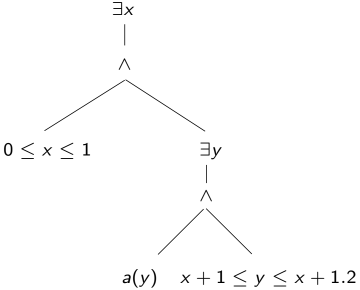


Eliminating a single top-level passive quantifier



Eliminating a single top-level passive quantifier: case 1





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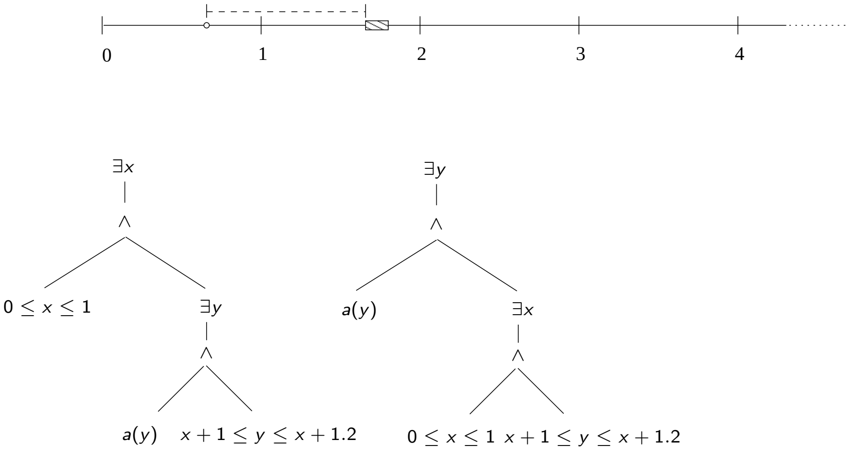

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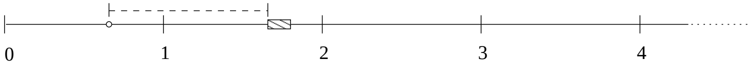

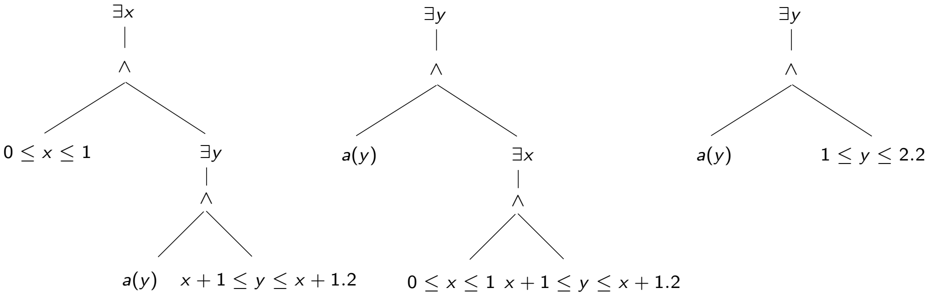

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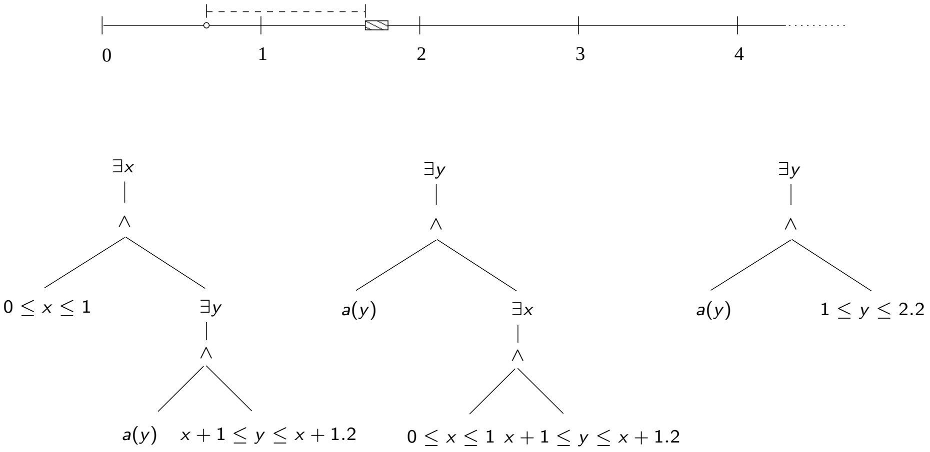
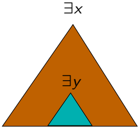


Interval constraint for x: $(0 \leq x \wedge y - 1.2 \leq x) \wedge (x \leq 1 \wedge x \leq y - 1)$.

Final step: Eliminating all passive quantifiers

Given an $\mathrm{FO}^c(<,+)$ sentence φ:

1. Convert to normal form.
2. While there is a passive quantifier node, repeat:
   - Pick a minimal such node.
   - Pull up $\vee's$ in its subtree (if any)
   - Now each disjunct is in $\exists$-normal form with single top-level passive quantifier. Eliminate this quantifer to get a disjunction of formulas in active normal form.



Summary

- Shown how to convert an $\mathrm{FO}^c(<,+)$ sentence to an equivalent actively quantified one.
- Gives us equivalence of pointwise and continuous semantics of $\mathrm{FO}(<,+)$.
- Equivalence of pointwise and continuous semantics of TPTL_S follows.
- Some open questions:
  - Compexity?!
  - What about TPTL (without "Since")?