Dualities and logical aspects of Baire functions

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It includes joint works with Antonio Di Nola, Giacomo Lenzi and Ioana Leuştean

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Summary

1. preliminary notions and background algebraic results
2. characterization of free objects
3. a duality

- Di Nola A., Lapenta S., Leuştean I., *An infinitary logic for basically disconnected compact Hausdorff spaces*, Journal of Logic and Computation, 28(6) (2018) 1275–1292.
- Di Nola A., Lapenta S., Lenzi G., *Dualities and algebraic geometry of Baire functions in Non-classical Logic*, submitted for publication.

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A is an MV-algebra iff $A \in HSP([0, 1])$

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Take (G, u) ℓu-group, u is strong order unit:

$$\text{for any } x \text{ exists } n \text{ such that } x \vee (-x) \leq nu$$

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$$([0, u]_G, \oplus, \neg, 0) \text{ is an MV-algebra}$$

Categorical equivalence. Mundici, 1986

For any MV-algebra A there exists (G, u) such that

$$A \simeq [0, u]_G$$

From MV-algebras to Riesz MV-algebras

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Riesz MV-algebras (Di Nola, Leuştean, 2014)

Some examples:

Any MV-algebra of $[0,1]$-valued function is a Riesz MV-algebra:

- $[0,1]^X$ for any nonempty X;
- $C(X)$, for any compact X.

Functional representation of the free Riesz MV-algebra

$f : [0,1]^n \to [0,1]$ is a $\mathrm{PWL}_u(\mathbb{R})$ function if it is continuous and there is a finite set of affine functions $p_1, \ldots, p_k : \mathbb{R}^n \to \mathbb{R}$ with coefficients in $\mathbb{R}$ such that for any $\mathrm{a} \in [0,1]^n$ there exists $i \in \{1, \ldots, k\}$ with $f(\mathrm{a}) = p_i(\mathrm{a})$.

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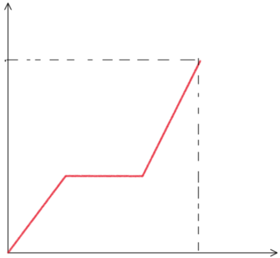

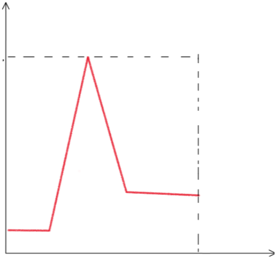

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$\mathrm{RMV}_\sigma = HSP([0,1])$

It is an infinitary variety of algebras! [Di Nola, Lapenta, Leuştean, 2018]

By standard results of UA, the free κ generated algebra is also the algebra of term functions build upon κ variables in the language of $\oplus, \neg, 0, \bigvee, \{\alpha\}_{\alpha\in[0,1]}$.

Free algebras in RMV_σ

Theorem (Di Nola, Lapenta, Lenzi)

The free κ-generated algebra in RMV_σ is the the algebra

$$Baire([0,1]^\kappa) = \{a : [0,1]^\kappa \to [0,1] \mid a \text{ is Baire-measurable}\},$$

generated by the projections π_i, $i \in \kappa$.

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- A Baire set in X is an element of the σ-algebra generated by the zerosets of continuous functions $f : X \to [0,1]$.
- A function $a : X \to Y$ is Baire measurable if preimages of Baire sets of Y are Baire sets of X.

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The free n-generated algebra in RMV_σ is the the algebra

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generated by $\pi_1, \ldots, \pi_n$.

- A Borel set in (X, τ) is an element of the σ-algebra generated by the open set of τ.
- A function $a : X \to Y$ is Borel measurable if preimages of Borel sets of Y are Borel sets of X.

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- A Borel set in (X, τ) is an element of the σ-algebra generated by the open set of τ.
- A function $a : X \to Y$ is Borel measurable if preimages of Borel sets of Y are Borel sets of X.
- In general $\mathcal{B}a(X) \subset \mathcal{B}o(X)$. When $\kappa \leq \omega$, $\mathrm{Baire}([0,1]^\kappa) = \mathrm{Borel}([0,1]^\kappa)$.

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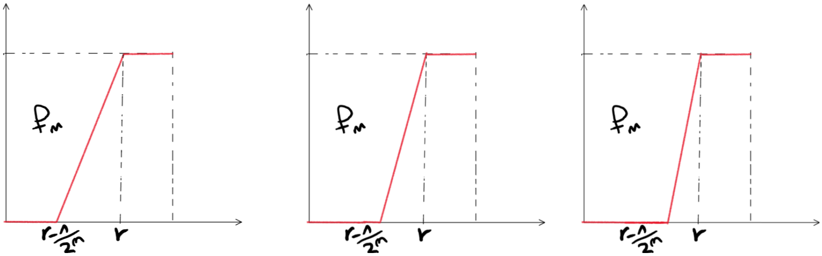

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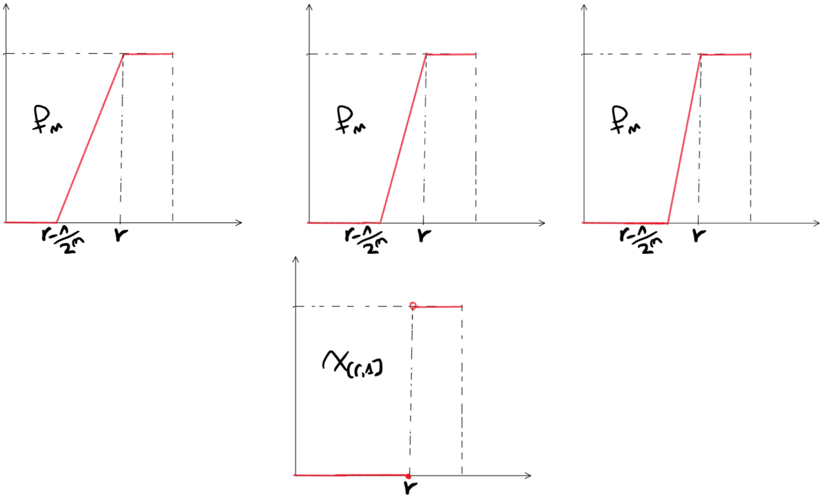

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- J is a σ-ideal of $\mathrm{Baire}([0,1]^X)$: it is closed under countable suprema!
- J is a set of Baire functions, but also term-functions, and "polynomials"

Operators on hypercubes and formulas

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For any subset $S \subseteq [0,1]^X$,

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The operators $\mathbb{I} - \mathbb{V}$ form a Galois connection

$$S \subseteq \mathbb{V}(J) \Leftrightarrow J \subseteq \mathbb{I}(S)$$

From a connection to a duality

- $\mathsf{RMV}_{\sigma}^{\mathbf{p}}$:
  - objects: pairs $(\text{Baire}([0,1]^X), I)$, where I is an ideal in the free algebra $\text{Baire}([0,1]^X)$.
  - morphism: $h : (\text{Baire}([0,1]^X), I) \to (\text{Baire}([0,1]^Y), J)$ is induced by a unique homomorphism $h^p : \text{Baire}([0,1]^X) \to \text{Baire}([0,1]^Y)$ such that $h^p(I) \subseteq J$.

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- IRL:
  - objects: subsets of hypercubes of type $[0,1]^\kappa$ which are intersection of Baire subsets,
  - arrows: tuple of term functions $\eta = (\eta_y)_{y \in Y} : S \subseteq [0,1]^X \to T \subseteq [0,1]^Y$, with $\eta_y \in \mathrm{Baire}([0,1]^X)$.

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we get this for free!

General affine adjunctions, Nullstellensätze, and dualities, Journal of Pure and Applied Algebra. Vol. 225. Pag.1-34

On objects, the adjunction is given by:

- $(\mathrm{Baire}([0,1]^X), J) \mapsto \mathbb{V}(J) \subseteq [0,1]^X$,
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Definition

$A \in \mathbf{RMV}_{\sigma}$ is called σ-semisimple iff
$\bigcap\{M \subseteq A \mid M \in Max(A) \cap Id_{\sigma}(A)\} = \{0\}$

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The duality further restricts

- Finitely presented algebras of RMV_σ correspond to Baire subsets of finite-dimensional hypercubes.
- Countably presented algebras of RMV_σ correspond to Baire subsets of the Hilbert cube.

Finally...

- We have a well behaved infinitary variety of algebras;

- There is a conservative extension of Łukasiewicz logic that has exactly these algebras as model. Such logic introduced in [Di Nola, Lapenta, Leuştean, 2018] and it is standard complete w.t.r. $[0, 1]$.

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- this variety has in its language (almost) all operations needed to discuss probability;
- Indeed, this work is an intermediate step towards a more ambitious metamathematics of probability

Thank you!