# Commonly Knowing Whether

J. Fan[1,2] D. Grossi[3,4] B. Kooi[3] X. Su[3] R. Verbrugge[3]

[1]Institute of Philosophy, Chinese Academy of Sciences, Beijing, China

[2]University of Chinese Academy of Sciences, Beijing, China

[3]University of Groningen, Groningen, Netherlands

[4]University of Amsterdam, Amsterdam, Netherlands

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# Outline

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- **Beyond 'knowing that'** ($K_i\varphi$)

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  - $Ew_1\varphi := E\varphi \vee E\neg\varphi$
  - $Ew_2\varphi := \bigwedge_{i\in \mathsf{G}} Kw_i\varphi$.
- Over $\mathcal{K}$-frames, $\models Ew_1\varphi \to Ew_2\varphi$.
- Over $\mathcal{T}$-frames, $\models Ew_1\varphi \leftrightarrow Ew_2\varphi$.

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- $\boldsymbol{Cw}_4\varphi := \bigwedge_{i\in\mathbf{G}}(\boldsymbol{CKw}_i\varphi \vee \boldsymbol{C}\neg\boldsymbol{Kw}_i\varphi)$
- $\boldsymbol{Cw}_5\varphi := \bigwedge_{i,j,k,\cdots\in\mathbf{G}}(\boldsymbol{Kw}_i\varphi \wedge \boldsymbol{Kw}_j\boldsymbol{Kw}_i\varphi \wedge \boldsymbol{Kw}_j\boldsymbol{Kw}_k\boldsymbol{Kw}_i\varphi \wedge \cdots)$

# Logical Relations over $\mathcal{K}$ and $\mathcal{KD}45$-frames

## Theorem

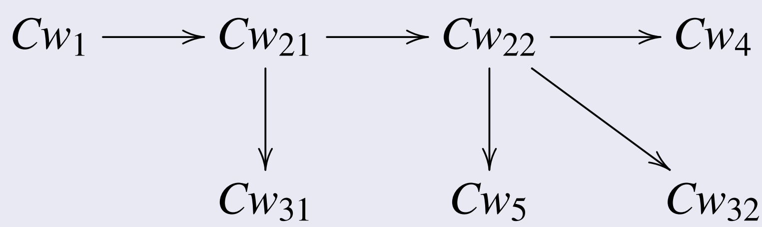


Figure 1: *Over $\mathcal{K}$ and $\mathcal{KD}45$-frames*

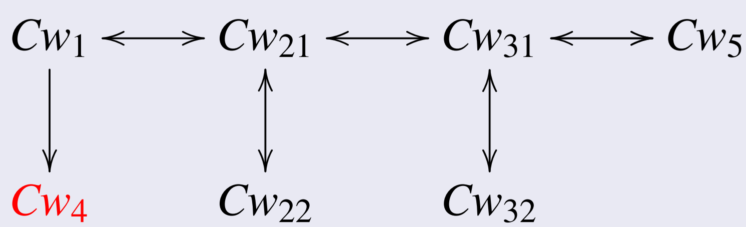


Figure 2: *Over $\mathcal{T}$ and $\mathcal{S}5$-frames*

# Axiomatization $\mathbb{C}w\mathbb{S}5$

## Definition

We fix a denumerable set of propositional atoms $\mathsf{P}$ and a nonempty finite set of agents $\mathsf{G}$. The language **Cw** can be defined by the following BNF:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid Kw_i\varphi \mid Cw\varphi,$$

where $p \in \mathsf{P}$ and $i \in \mathsf{G}$.

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## Definition

$\mathcal{M}, w \vDash Kw_i\varphi$ iff $\mathcal{M}, w \vDash K_i\varphi$ or $\mathcal{M}, w \vDash K_i\neg\varphi$.

$\mathcal{M}, w \vDash Cw\varphi$ iff $\mathcal{M}, w \vDash C\varphi$ or $\mathcal{M}, w \vDash C\neg\varphi$.

# Axiomatization $\mathbb{C}w\mathbb{S}5$

| | |
|---|---|
| (TAUT) | All instances of tautologies |
| (Kw-DIS) | $Kw_i\varphi \to Kw_i(\varphi \to \psi) \vee Kw_i(\neg\varphi \to \chi)$ |
| (Kw-CON) | $Kw_i(\chi \to \varphi) \wedge Kw_i(\neg\chi \to \varphi) \to Kw_i\varphi$ |
| (Kw-T) | $Kw_i\varphi \wedge Kw_i(\varphi \to \psi) \wedge \varphi \to Kw_i\psi$ |
| (wKw-5) | $\neg Kw_i\varphi \to Kw_i\neg Kw_i\varphi$ |
| (Kw-$\leftrightarrow$) | $Kw_i\varphi \leftrightarrow Kw_i\neg\varphi$ |
| (Cw-DIS) | $Cw\varphi \to Cw(\varphi \to \psi) \vee Cw(\neg\varphi \to \chi)$ |
| (Cw-CON) | $Cw(\chi \to \varphi) \wedge Cw(\neg\chi \to \varphi) \to Cw\varphi$ |
| (Cw-$\leftrightarrow$) | $Cw\varphi \leftrightarrow Cw\neg\varphi$ |
| (Cw-T) | $Cw\varphi \wedge Cw(\varphi \to \psi) \wedge \varphi \to Cw\psi$ |
| (Cw-Ind) | $Cw(\varphi \to Ew\varphi) \to (\varphi \to Cw\varphi)$ |
| (Cw-Mix) | $Cw\varphi \to Ew\varphi \wedge EwCw\varphi$ |
| (Kw-NEC) | from $\varphi$ infer $Kw_i\varphi$ |
| (C-NEC) | from $\varphi$ infer $Cw\varphi$ |
| (Kw-RE) | from $\varphi \leftrightarrow \psi$ infer $Kw_i\varphi \leftrightarrow Kw_i\psi$ |
| (Cw-RE) | from $\varphi \leftrightarrow \psi$ infer $Cw\varphi \leftrightarrow Cw\psi$ |
| (MP) | from $\varphi$ and $\varphi \to \psi$ infer $\psi$ |

# Completeness

The basic idea of the completeness proof:

$\mathbb{C}w\mathbb{S}5$ is not compact

↓

weak completeness

↓

construct finite canonical models

↓

define finite closure for each $\varphi$ $\star$

**Theorem**

*The logic $\mathbb{C}w\mathbb{S}5$ is weakly complete with respect to $\mathcal{S}5$.*

# Expressivity of $Cw_5$

- Focus on $Cw_5 := \bigwedge_{i,j,k,\dots \in \mathsf{G}}(Kw_i\varphi \wedge Kw_jKw_i\varphi \wedge Kw_jKw_kKw_i\varphi \wedge \cdots)$

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- Compare the expressivity of $\mathbf{Cw_5}$ with $\mathbf{C}$:

$$\begin{array}{ll} \mathbf{Cw_5} & \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid Kw_i\varphi \mid Cw_5\varphi \\ \mathbf{C} & \varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_i\varphi \mid C\varphi \end{array}$$

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$Kw_i\varphi = K_i\varphi \vee K_i\neg\varphi$ ——— $Cw_5\varphi = Kw_i\varphi \wedge Kw_jKw_i\varphi \wedge \cdots$

↓ *guess*

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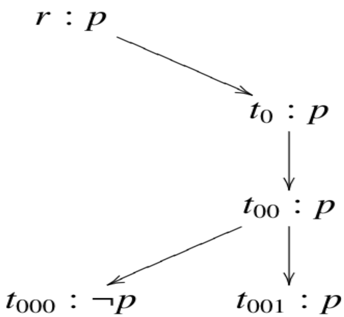


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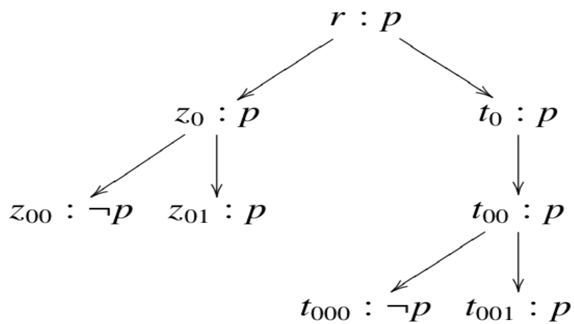


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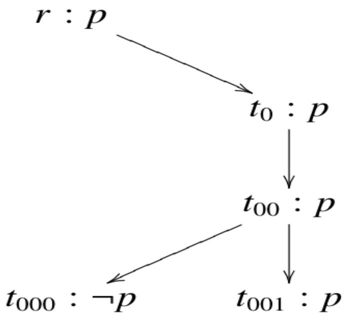


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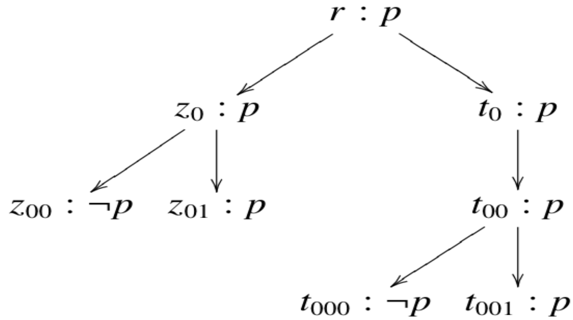


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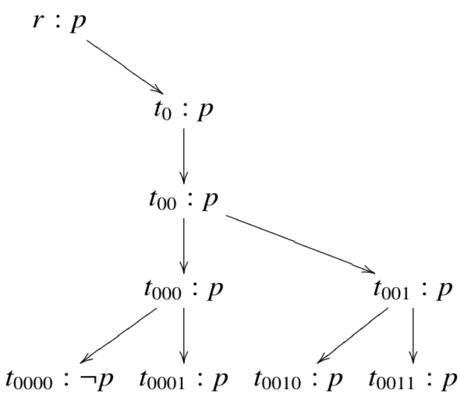
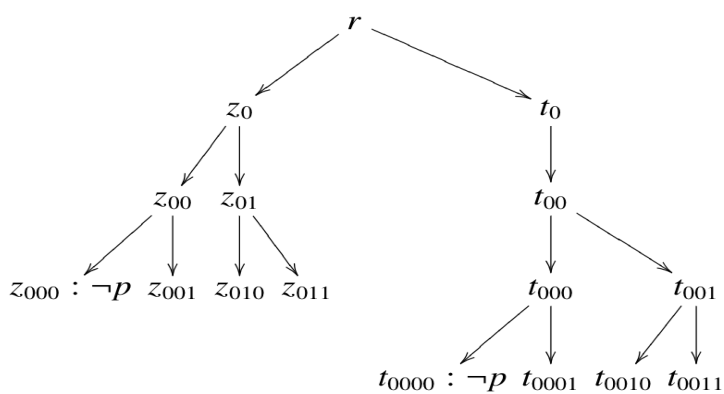



$M_2$

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**Lemma**

1. *Over $\mathcal{K}$, $\mathbf{Cw_5}$ is not weaker in expressivity than $\mathbf{C}$.*
2. *Over $\mathcal{K}$, $\mathbf{Cw_5}$ and $\mathbf{C}$ are incomparable in expressivity.*

# Conclusions

1. Five possible notions of ‘commonly knowing whether’;
2. On $\mathcal{S}5$-frames four of the five notions boil down to the same thing;
3. Soundness and weak completeness of $\mathbb{C}w\mathbb{S}5$ over $\mathcal{S}5$-frames;
4. Over $\mathcal{K}$, $\mathbf{Cw}_5$ and $\mathbf{C}$ are incomparable in expressivity.