Double Boolean Algebras with Operators

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Overview

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- $\mathfrak{H}(\mathbb{K}) \subseteq \mathfrak{P}(\mathbb{K})$.

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- For all $x, y \in \mathfrak{P}(\mathbb{K})$: $x \leq y$ if and only if $x \sqcap y = x \sqcap x$ and $x \sqcup y = y \sqcup y$.

Double Boolean algebra [3]

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Definition (2)

A *double Boolean algebra* (dBa) $\mathbf{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ is an abstract algebra which satisfies all the following equations:

1a. $(x \sqcap x) \sqcap y = x \sqcap y$
2a. $x \sqcap y = y \sqcap x$
3a. $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$
4a. $\neg(x \sqcap x) = \neg x$
5a. $x \sqcap (x \sqcup y) = x \sqcap x$
6a. $x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z)$
7a. $x \sqcap (x \vee y) = x \sqcap x$
8a. $\neg\neg(x \sqcap y) = x \sqcap y$

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4b. $\lrcorner(x \sqcup x) = \lrcorner x$
5b. $x \sqcup (x \sqcap y) = x \sqcup x$
6b. $x \sqcup (y \wedge z) = (x \sqcup y) \wedge (x \sqcup z)$
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$x \vee y := \neg(\neg x \sqcap \neg y)$, and $x \wedge y := \lrcorner(\lrcorner x \sqcup \lrcorner y)$, for all $x, y \in D$.

Double Boolean Algebra.

Continuation Of definition (2)

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9b. $x \sqcup \lrcorner x = \top$

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  - $\sqsubseteq$ is shown to be a quasi-order on **D**.

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  - For $B(\subseteq M) \rightarrow (B', B)$.

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  - $\overline{A}^E := \{x \in W \text{ such that } E(x) \cap A \neq \emptyset\}$.

Proposition

- $\overline{A} = (\underline{(A^c)})^c, \underline{A} = (\overline{(A^c)})^c$.
- $\underline{W} = W$.
- $\underline{A \cap B} = \underline{A} \cap \underline{B}, \overline{A \cup B} = \overline{A} \cup \overline{B}$.
- $A \subseteq B$ implies that $\underline{A} \subseteq \underline{B}, \overline{A} \subseteq \overline{B}$.

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2a	$I(\neg\bot) = \neg\bot$	2b	$C(\lrcorner\top) = \lrcorner\top$
3a	$I(x \sqcap x) = I(x)$	3b	$C(x \sqcup x) = C(x)$

- Any Boolean algebra with operator(Bao) is a dBao.

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Theorem

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- $\underline{\mathfrak{H}}^+(\mathbb{KC})$ is an example of a *pure dBao* (a dBao based on a pure dBa).

Future Work

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- The algebraic semantics and a semantics based on the class of Kripke contexts.

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Thank You