Quotient Models in a Class of Non-Classical Set Theories

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2. For any ordinal α we define,

$$\mathsf{V}_{\alpha}^{(\mathbb{B})} = \{x : \mathrm{Func}(x) \wedge \mathrm{ran}(x) \subseteq B \wedge \exists \xi < \alpha (\mathrm{dom}(x) \subseteq \mathsf{V}_{\xi}^{(\mathbb{B})})\}$$

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3. Using the above we get a Boolean valued model as,

$$\mathsf{V}^{(\mathbb{B})} = \{x : \exists \alpha (x \in \mathsf{V}_\alpha^{(\mathbb{B})})\}$$

4. Extend the language of classical ZFC by adding a name corresponding to each element of $\mathsf{V}^{(\mathbb{B})}$, in it.
5. Associate every formula of the extended language with a value of B by the map $[\![.]\!]$. First we give the algebraic expressions which associate the two basic well-formed formulas with values of B. For any u, v in $\mathsf{V}^{(\mathbb{B})}$,

$$[\![u \in v]\!] = \bigvee_{x \in \mathrm{dom}(v)} (v(x) \wedge [\![x = u]\!])$$

$$[\![u = v]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![x \in v]\!]) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow [\![y \in u]\!])$$

6. Then for any sentences σ and τ of the new language we define,

$$\llbracket \sigma \wedge \tau \rrbracket = \llbracket \sigma \rrbracket \wedge \llbracket \tau \rrbracket$$
$$\llbracket \sigma \vee \tau \rrbracket = \llbracket \sigma \rrbracket \vee \llbracket \tau \rrbracket$$
$$\llbracket \sigma \rightarrow \tau \rrbracket = \llbracket \sigma \rrbracket \Rightarrow \llbracket \tau \rrbracket$$
$$\llbracket \neg \sigma \rrbracket = \llbracket \sigma \rrbracket^*$$
$$\llbracket \forall x \varphi(x) \rrbracket = \bigwedge_{x \in V^{(\mathbb{B})}} \llbracket \varphi(x) \rrbracket$$
$$\llbracket \exists x \varphi(x) \rrbracket = \bigvee_{x \in V^{(\mathbb{B})}} \llbracket \varphi(x) \rrbracket$$

7. A sentence σ will be called *valid* in $V^{(\mathbb{B})}$ or $V^{(\mathbb{B})}$ will be called a model of a sentence σ if $\llbracket \sigma \rrbracket = 1$. It will be denoted as $V^{(\mathbb{B})} \models \sigma$.

8. Then we have the following celebrated result:

Theorem

For any complete Boolean algebra $\mathbb{B}$, $\mathrm{V}^{(\mathbb{B})} \models \mathrm{ZFC}$, *i.e., all the classical logic axioms and* ZFC *axioms are valid in* $\mathrm{V}^{(\mathbb{B})}$.

Reasonable Implication Algebra-Valued Models

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If $\mathbb{A}$ is a deductive reasonable implication algebra and D is any designated set then $\mathsf{V}^{(\mathbb{A})} \models_D \mathrm{NFF\text{-}ZF}^-$.

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A three-valued deductive reasonable implication algebra PS_3 was produced, whose logic is non-classical, in particular paraconsistent. Hence, as an application of the above theorem, we get $\mathsf{V}^{(\mathrm{PS}_3)} \models_D \mathrm{NFF\text{-}ZF}$, i.e., $\mathsf{V}^{(\mathrm{PS}_3)}$ becomes a model of non-classical set theory.

Failure of Leibniz's Law in $\mathrm{V}^{(\mathrm{PS}_3)}$

Let us define a class relation $\sim$ in $\mathrm{V}^{(\mathrm{PS}_3)}$ as $u \sim v$ iff $\mathrm{V}^{(\mathrm{PS}_3)} \models u = v$, where $u, v \in \mathrm{V}^{(\mathrm{PS}_3)}$.

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Let us define a class relation $\sim$ in $V^{(PS_3)}$ as $u \sim v$ iff $V^{(PS_3)} \models u = v$, where $u, v \in V^{(PS_3)}$. It can be proved that $\sim$ is an equivalence relation and hence we have the quotient space $V^{(PS_3)}/\sim$.

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Question: How effective this quotient space is?

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Question: How effective this quotient space is?
Answer: Not much, as $V^{(PS_3)}$ does not satisfy the *Leibniz's law of indiscernibility of identicals:*

$$\forall x \forall y \big((x = y \wedge \varphi(x)) \rightarrow \varphi(y)\big) \qquad (\mathrm{LL}_\varphi)$$

for all formulas $\varphi(x)$.

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(ii) $\mathbb{A}$ has a unique atom and a unique co-atom,

(iii) the two algebraic operators $\Rightarrow$ and * are defined as follows:

$$a \Rightarrow b = \begin{cases} 0, & \text{if } a \neq 0 \text{ and } b = 0; \\ 1, & \text{otherwise}; \end{cases}$$

$$a^* = \begin{cases} 0, & \text{if } a = 1; \\ a, & \text{if } a \in D \setminus \{1\}; \\ 1, & \text{if } a \notin D. \end{cases}$$

The $[\![\cdot]\!]_{\mathrm{PA}}$-Assignment Function

Let $(\mathbb{A}, D)$ be an MTV-algebra. We replace the usual assignment function $[\![\cdot]\!]$ by a new assignment function $[\![\cdot]\!]_{\mathrm{PA}}$, where the algebraic expressions of both the assignment functions are same except the expressions of the atomic formulas with equality: for any $u, v \in \mathrm{V}^{(\mathbb{A})}$,

$$[\![u = v]\!]_{\mathrm{PA}} = \bigwedge_{x \in \mathrm{dom}(u)} \left((u(x) \Rightarrow [\![x \in v]\!]_{\mathrm{PA}}) \wedge ([\![x \in v]\!]^*_{\mathrm{PA}} \Rightarrow u(x)^*)\right)$$

$$\wedge \bigwedge_{y \in \mathrm{dom}(v)} \left((v(y) \Rightarrow [\![y \in u]\!]_{\mathrm{PA}}) \wedge ([\![y \in u]\!]^*_{\mathrm{PA}} \Rightarrow v(y)^*)\right).$$

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For any MTV-algebra $(\mathbb{A}, D)$ and any sentence φ, we say that φ is valid in $\mathsf{V}^{(\mathbb{A},\ [\![\cdot]\!]_{\mathrm{PA}})}$, denoted by $\mathsf{V}^{(\mathbb{A},\ [\![\cdot]\!]_{\mathrm{PA}})} \models_D \varphi$ iff $[\![\varphi]\!]_{\mathrm{PA}} \in D$.

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Answer: No; the Axiom of Extensionality fails.

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Thank You...