Relative Expressive Powers of First Order Modal Logic and Term Modal Logic

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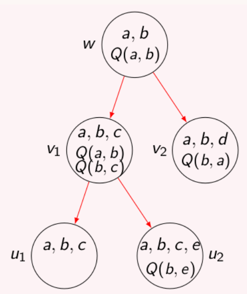
First Order Modal Logic (FOML)

Syntax

$$\alpha := Q(\overline{x}) \mid \neg\alpha \mid \alpha\wedge\alpha \mid \exists x\ \alpha \mid \Diamond\alpha$$

Example: $\forall x \exists y \Box Q(x,y)$



$\mathcal{M}^\mathcal{F} = (\mathcal{W}, \mathcal{D}, \delta, \mathcal{R}^\mathcal{F}, \rho)$ where:

- $\mathcal{W}$ (world set)
- $\mathcal{D}$ (domain set)
- $\delta : \mathcal{W} \mapsto 2^\mathcal{D}$ (local domain set)
- $\mathcal{R}^\mathcal{F} \subseteq (\mathcal{W}\times\mathcal{W})$
- $\rho : (\mathcal{W}\times\mathcal{P}) \mapsto \bigcup_k 2^{\mathcal{D}^k}$

$\mathcal{M}^\mathcal{F}, w, \sigma \models Q(\overline{x}) \quad \Leftrightarrow (\sigma(\overline{x})) \in \rho(w, Q)$

$\mathcal{M}^\mathcal{F}, w, \sigma \models \neg\alpha \quad \Leftrightarrow \mathcal{M}^\mathcal{F}, w, \sigma \not\models \alpha$

$\mathcal{M}^\mathcal{F}, w, \sigma \models (\alpha\wedge\beta) \Leftrightarrow \mathcal{M}^\mathcal{F}, w, \sigma \models \alpha$ and $\mathcal{M}^\mathcal{F}, w, \sigma \models \beta$

$\mathcal{M}^\mathcal{F}, w, \sigma \models \exists x\ \alpha \quad \Leftrightarrow$ there is some $d \in \delta(w)$ such that $\mathcal{M}^\mathcal{F}, w, \sigma_{[x\mapsto d]} \models \alpha$

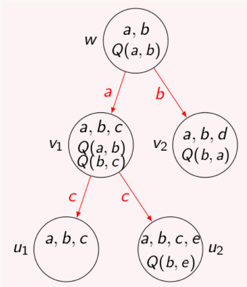
$\mathcal{M}^\mathcal{F}, w, \sigma \models \Diamond\ \alpha \quad \Leftrightarrow$ there is some $u \in \mathcal{W}$ such that $(w,u) \in \mathcal{R}^\mathcal{F}$ and $\mathcal{M}^\mathcal{F}, u, \sigma \models \alpha$

Term Modal Logic (TML)

Syntax

$$\phi := Q(\overline{x}) \mid \neg\phi \mid \phi\wedge\phi \mid \exists x\ \phi \mid \Diamond_x\phi$$

Example: $\forall x \exists y \Box_x Q(x,y)$



$\mathcal{M}^\mathcal{T} = (\mathcal{W}, \mathcal{D}, \delta, \mathcal{R}^\mathcal{T}, \rho)$ where:

- $\mathcal{W}$ (world set)
- $\mathcal{D}$ (agent set)
- $\delta : \mathcal{W} \mapsto 2^\mathcal{D}$ (local agent set)
- $\mathcal{R}^\mathcal{T} \subseteq (\mathcal{W}\times\mathcal{D}\times\mathcal{W})$
- $\rho : (\mathcal{W}\times\mathcal{P}) \mapsto \bigcup_k 2^{\mathcal{D}^k}$

$\mathcal{M}^\mathcal{T}, w, \sigma \models Q(\overline{x}) \quad \Leftrightarrow (\sigma(\overline{x})) \in \rho(w, Q)$

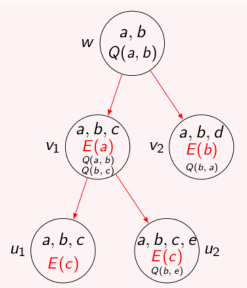
$\mathcal{M}^\mathcal{T}, w, \sigma \models \neg\phi \quad \Leftrightarrow \mathcal{M}^\mathcal{T}, w, \sigma \not\models \phi$

$\mathcal{M}^\mathcal{T}, w, \sigma \models \phi\wedge\psi \Leftrightarrow \mathcal{M}^\mathcal{T}, w, \sigma \models \phi$ and $\mathcal{M}^\mathcal{T}, w, \sigma \models \psi$

$\mathcal{M}^\mathcal{T}, w, \sigma \models \exists x\ \phi \quad \Leftrightarrow$ there is some $d \in \delta(w)$ such that $\mathcal{M}^\mathcal{T}, w, \sigma_{[x\mapsto d]} \models \phi$

$\mathcal{M}^\mathcal{T}, w, \sigma \models \Diamond_x\ \phi \quad \Leftrightarrow$ there is some $u \in \mathcal{W}$ such that $(w, \sigma(x), u) \in \mathcal{R}^\mathcal{T}$ and $\mathcal{M}^\mathcal{T}, u, \sigma \models \phi$

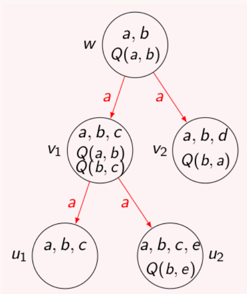
$\mathrm{TML}^k \subseteq \mathrm{FOML}^k$ (over $\mathcal{K}$-frames)



- $\mathsf{Tr}_1(Q(\overline{x})) := Q(\overline{x})$
- $\mathsf{Tr}_1(\neg\phi) := \neg\,\mathsf{Tr}_1(\phi)$
- $\mathsf{Tr}_1(\alpha\wedge\psi) := \mathsf{Tr}_1(\phi)\wedge\mathsf{Tr}_1(\psi)$
- $\mathsf{Tr}_1(\exists x\ \phi) := \exists x\ \mathsf{Tr}_1(\phi)$
- $\mathsf{Tr}_1(\Diamond_x\phi) := \Diamond\big(E(x)\wedge\mathsf{Tr}_1(\phi)\big)$

$$\mathcal{M}^\mathcal{T}, w, \sigma \models \phi \text{ iff } \mu(\mathcal{M}^\mathcal{T}), w, \sigma \models \mathsf{Tr}_1(\phi)$$

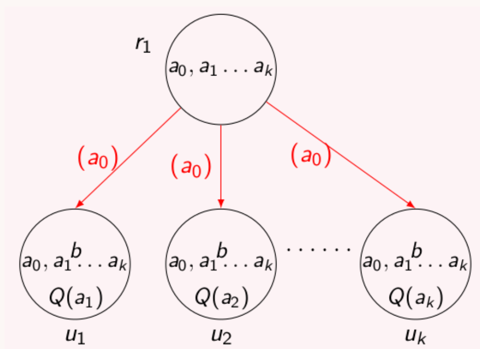
$\mathrm{FOML}^k \subseteq \mathrm{TML}^{k+1}$ (over $\mathcal{K}$-frames)

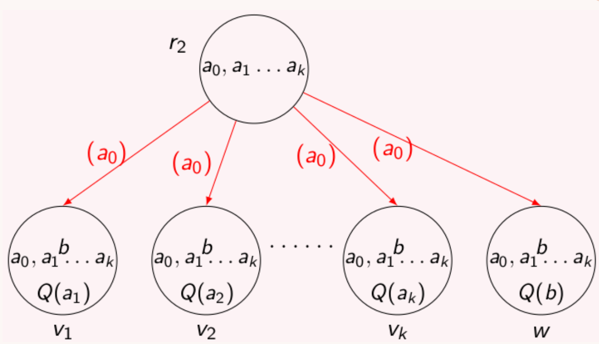


- $\mathsf{Tr}_2(Q(\overline{x})) := Q(\overline{x})$
- $\mathsf{Tr}_2(\neg\alpha) := \neg\,\mathsf{Tr}_2(\alpha)$
- $\mathsf{Tr}_2(\alpha\wedge\beta) := \mathsf{Tr}_2(\alpha)\wedge\mathsf{Tr}_2(\beta)$
- $\mathsf{Tr}_2(\exists x\ \alpha) := \exists x\ \mathsf{Tr}_2(\alpha)$
- $\mathsf{Tr}_2(\Diamond\alpha) := \Diamond_z\ (\mathsf{Tr}_2(\alpha))$ where z is fresh

$$\mathcal{M}^\mathcal{F}, w, \sigma \models \alpha \text{ iff } \pi(\mathcal{M}^\mathcal{F}), w, \sigma_{[z\to a]} \models \mathsf{Tr}_2(\alpha)$$

$\mathrm{FOML}^k \not\subseteq \mathrm{TML}^k$ (over $\mathcal{K}$-frames)





$$\mathcal{M}^\mathcal{F}{}_1, r_1 \models \exists x_1\ldots x_k \Box(\bigvee_i Q(x_i)) \qquad \mathcal{M}^\mathcal{F}{}_2, r_2 \not\models \exists x_1\ldots x_k \Box(\bigvee_i Q(x_i))$$

For every k-variable TML formula ϕ: $\pi(\mathcal{M}^\mathcal{F}{}_1), r_1, \sigma \models \phi$ iff $\pi(\mathcal{M}^\mathcal{F}{}_2), r_2, \sigma \models \phi$