On the Characterizations of Tarski-type and Lindenbaum-type Logical Structures

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Historical Background

- **Tarski's theory of consequence operators**
  - The structures he considered essentially are pairs of the form $(\mathscr{L}, \vdash)$ where $\mathscr{L}$ is a **set** and $\vdash \subseteq \mathcal{P}(\mathscr{L}) \times \mathscr{L}$ **satisfying certain properties**.

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- **Polish Logic**
  - The structures they considered essentially are pairs of the form $(\mathscr{L}, \vdash)$ where $\mathscr{L}$ is an **absolutely free algebra** and $\vdash \subseteq \mathcal{P}(\mathscr{L}) \times \mathscr{L}$ **satisfying certain properties**.

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- **Suszko's Abstract Logic**
  - The structures considered by Suszko and his collaborators essentially are pairs of the form $(\mathscr{L}, \vdash)$ where $\mathscr{L}$ is an **algebra** and $\vdash \subseteq \mathcal{P}(\mathscr{L}) \times \mathscr{L}$ **satisfying certain properties**.

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- **Béziau's Universal Logic**
  - Universal logic was defined by Béziau to be the study of pairs of the form $(\mathscr{L}, \vdash)$ where $\mathscr{L}$ is a **set** and $\vdash \subseteq \mathcal{P}(\mathscr{L}) \times \mathscr{L}$.

Definitions

Definition 2.1 (Logical Structures)

A *logical structure* is a pair of the form $(\mathscr{L}, \vdash)$ where $\mathscr{L}$ is a set and $\vdash \subseteq \mathcal{P}(\mathscr{L}) \times \mathscr{L}$.

Example 1

The pair $(\mathbf{CPC}, \vdash_{\mathbf{CPC}})$ where,

- $\mathbf{CPC}$ denotes the set of all wffs of classical propositional logic and,
- $\vdash_{\mathbf{CPC}}$ denotes the usual proof theoretic consequence relation.

is a logical structure.

Definitions

Definition 2.2 (Tarski-type Logical Structures)

A *logical structure* $(\mathscr{L}, \vdash)$ is called *Tarski-type logical structure* (or simply, *Tarski-type*) if $\vdash$ satisfies the following properties:

(a) For all $\Gamma \subseteq \mathscr{L}$ and all $\alpha \in \mathscr{L}$, if $\alpha \in \Gamma$ then $\Gamma \vdash \alpha$. (Reflexivity)

(b) For all $\Gamma, \Sigma \subseteq \mathscr{L}$ and all $\alpha \in \mathscr{L}$, if $\Gamma \vdash \alpha$ and $\Gamma \subseteq \Sigma$ then $\Sigma \vdash \alpha$. (Monotonicity)

(c) For all $\Gamma, \Sigma \subseteq \mathscr{L}$ and all $\alpha \in \mathscr{L}$, if $\Gamma \vdash \beta$ for all $\beta \in \Sigma$ then, $\Sigma \vdash \alpha$ implies that $\Gamma \vdash \alpha$. (Transitivity)

Example 2

$(\mathbf{CPC}, \vdash_{\mathbf{CPC}})$ as previously defined is a Tarski-type logical structure.

Definitions

Definition 2.3 (Deductively Closed Sets)

Let $(\mathscr{L}, \vdash)$ be a logical structure and $\Sigma \subseteq \mathscr{L}$. Then Σ is called a *deductively closed set* in $\mathscr{L}$ if it satisfies the following properties.

(1) For all $\Gamma \subseteq \Sigma$ and all $\beta \in \mathscr{L}$, if $\Gamma \vdash \beta$ then $\beta \in \Sigma$.

(2) For all $\beta \in \Sigma$, $\Sigma \vdash \beta$.

Example 3

In $(\mathbf{CPC}, \vdash_{\mathbf{CPC}})$ the set of theorems is a deductively closed set.

Definitions

Definition 2.4 (Strongly-Lindenbaum-Type Logical Structures)

A logical structure $(\mathscr{L}, \vdash)$ is called a *strongly-Lindenbaum-type logical structure* (or, simply *strongly-Lindenbaum-type*) if for all $\Gamma \subseteq \mathscr{L}$ and all $\alpha \in \mathscr{L}$ the set,

$$\mathtt{T}_{\alpha}^{\Gamma} := \{\Sigma : \Gamma \subseteq \Sigma \text{ and } \Sigma \nvdash \alpha\}$$

has a maximal element whenever it is non-empty.

Example 4

$(\mathbf{CPC}, \vdash_{\mathbf{CPC}})$ as previously defined is a strongly-Lindenbaum-type logical structure.

Definitions

Definition 2.5 (α-saturated sets)

Let $(\mathscr{L}, \vdash)$ be a logical structure. Let $\Gamma \subseteq \mathscr{L}$ and $\alpha \in \mathscr{L}$. Then Γ is called α-saturated in $\mathscr{L}$ if $\Gamma \nvdash \alpha$ and for all $\beta \in \mathscr{L} \setminus \Gamma$, $\Gamma \cup \{\beta\} \vdash \alpha$.

Example 5

In $(\mathbf{CPC}, \vdash_{\mathbf{CPC}})$ a maximal consistent set Δ is α-saturated iff $\Delta \nvdash_{\mathbf{CPC}} \alpha$.

Theorems

Theorem 3.1 (Characterization of Tarski-type Logical Structures)

Let $(\mathscr{L}, \vdash)$ be a logical structure. The following statements are equivalent,

(1) $(\mathscr{L}, \vdash)$ *is a Tarski-type Logic.*

(2) *For all $\Gamma \subseteq \mathscr{L}$ and for all $\alpha \in \mathscr{L}$ such that $\Gamma \nvdash \alpha$, there exists a deductively closed $\Sigma \subseteq \mathscr{L}$ such that $\Gamma \subseteq \Sigma$ and $\Sigma \nvdash \alpha$.*

Theorems

Theorem 3.2 (Characterization of Strongly Lindenbaum-type Logical Structures)

Let $(\mathscr{L}, \vdash)$ be a logical structure. Then the following statements are equivalent.

(1) *For all $\Gamma \subseteq \mathscr{L}$ and all $\alpha \in \mathscr{L}$, if $\Gamma \nvdash \alpha$ then there exists a maximal α-saturated set Σ such that $\Gamma \subseteq \Sigma$.*

(2) *$(\mathscr{L}, \vdash)$ is strongly-Lindenbaum-type.*

Theorems

Theorem 3.2 (Characterization of Strongly Lindenbaum-type Logical Structures)

Let $(\mathscr{L}, \vdash)$ be a logical structure. Then the following statements are equivalent.

(1) *For all $\Gamma \subseteq \mathscr{L}$ and all $\alpha \in \mathscr{L}$, if $\Gamma \nvdash \alpha$ then there exists a maximal α-saturated set Σ such that $\Gamma \subseteq \Sigma$.*

(2) *$(\mathscr{L}, \vdash)$ is strongly-Lindenbaum-type.*

Note:

- For $(\mathbf{CPC}, \vdash_{\mathbf{CPC}})$ the α-saturated sets correspond to maximal consistent sets of wffs.

Theorems

Theorem 3.2 (Characterization of Strongly Lindenbaum-type Logical Structures)

Let $(\mathscr{L}, \vdash)$ be a logical structure. Then the following statements are equivalent.

(1) *For all $\Gamma \subseteq \mathscr{L}$ and all $\alpha \in \mathscr{L}$, if $\Gamma \nvdash \alpha$ then there exists a maximal α-saturated set Σ such that $\Gamma \subseteq \Sigma$.*

(2) *$(\mathscr{L}, \vdash)$ is strongly-Lindenbaum-type.*

Note:

- For $(\mathbf{CPC}, \vdash_{\mathbf{CPC}})$ the α-saturated sets correspond to maximal consistent sets of wffs.
- Consequently, for $(\mathbf{CPC}, \vdash_{\mathbf{CPC}})$, (1) of Theorem 3.2 becomes the usual Lindenbaum Lemma.

Theorems

Theorem 3.3 (Characterization of TsL Logical Structures)

Let $(\mathscr{L}, \vdash)$ be a logical structure. Then the following statements are equivalent.

(1) *For all $\Gamma \subseteq \mathscr{L}$ and all $\alpha \in \mathscr{L}$, there exists a deductively closed α-saturated set $\Sigma \subseteq \mathscr{L}$ such that $\Gamma \subseteq \Sigma$.*

(2) *$(\mathscr{L}, \vdash)$ is both Tarski-type and strongly-Lindenbaum-type.*

Tarski-type $\overset{?}{\Longrightarrow}$ strongly-Lindenbaum-type

Tarski-type $\not\Rightarrow$ strongly-Lindenbaum-type

Let X be any infinite set.

Tarski-type $\overset{?}{\Longrightarrow}$ strongly-Lindenbaum-type

Tarski-type $\not\Rightarrow$ strongly-Lindenbaum-type

Let X be any infinite set. Define $C : \mathcal{P}(X) \to \mathcal{P}(X)$ as follows:

$$C(\Gamma) := \begin{cases} \Gamma & \text{if } \Gamma \text{ is finite} \\ X & \text{otherwise} \end{cases}$$

for all $\Gamma \subseteq X$.

Tarski-type $\overset{?}{\Longrightarrow}$ strongly-Lindenbaum-type

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for all $\Gamma \subseteq X$. Now let $\vdash_C$ be defined as follows:

$$\text{For all } \Gamma \subseteq X \text{ and } \alpha \in X,\ \Gamma \vdash_C \alpha \text{ iff } \alpha \in C(\Gamma)$$

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$$\text{For all } \Gamma \subseteq X \text{ and } \alpha \in X,\ \Gamma \vdash_C \alpha \text{ iff } \alpha \in C(\Gamma)$$

Then $(X, \vdash_C)$ is Tarski-type but not strongly-Lindenbaum-type.

We point out that Kuratowski closure operators corresponding to the cofinite topology on an infinite set always satisfies the above property.

Strongly-Lindenbaum-type $\overset{?}{\Longrightarrow}$ Tarski-type

Strongly-Lindenbaum-type $\not\Rightarrow$ Tarski-type

Consider $(\mathbf{CPC}, \vdash_{\mathbf{CPC}})$ again.

Strongly-Lindenbaum-type $\overset{?}{\Longrightarrow}$ Tarski-type

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Consider $(\mathbf{CPC}, \vdash_{\mathbf{CPC}})$ again. Define a new logical structure, say, $(\mathbf{CPC}, \models)$ as follows:

$$\text{For all } \Gamma \subseteq \mathbf{CPC} \text{ and all } \alpha \in \mathbf{CPC},\ \Gamma \models \alpha \text{ iff } \Gamma \setminus \{\alpha\} \vdash_{\mathbf{CPC}} \alpha$$

Strongly-Lindenbaum-type $\overset{?}{\Longrightarrow}$ Tarski-type

Strongly-Lindenbaum-type $\not\Rightarrow$ Tarski-type

Consider $(\mathbf{CPC}, \vdash_{\mathbf{CPC}})$ again. Define a new logical structure, say, $(\mathbf{CPC}, \models)$ as follows:

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Then $(\mathbf{CPC}, \models)$ is strongly-Lindenbaum-type but $\models$ is not reflexive. So $(\mathbf{CPC}, \models)$ is not Tarski-type.

Future Work

- Develop a theory of logical structures.
- Develop the graded counterparts of the notions talked about here and find their applications to the Graded Consequence Theory.

Thank You!