Feferman-Vaught decompositions for prefix classes of first order logic

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Introduction

- The Feferman-Vaught (FV) theorem from model theory gives a method to evaluate a first order (FO) sentence on a disjoint union of structures by providing other FO sentences to evaluate on the individual structures, and combining the results of the evaluations using a propositional formula.
- Historically: First shown for direct products (Mostowski, 1952) and later for generalized products (Feferman-Vaught, 1967)
- Numerous applications in computer science and finite model theory: decidability of theories, satisfiability checking, preservation theorems, algorithmic metatheorems
- FV decompositions over disjoint union for a sentence φ can be non-elementarily larger than φ.
- In special cases, can be computed in elementary time: Bounded degree structures and full FO (3-fold exp); FO$[2]$ and all structures (2-fold exp.)

Tree generalization of prenex formulae: $\mathrm{T}\Sigma_n$ and $\mathrm{T}\Pi_n$

- Let Σ_n, resp. Π_n = FO formulae in prenex normal form (PNF) with n quantifier blocks beginning with an $\exists$ block, resp. $\forall$ block. The quantifier-free parts are assumed to be in negation normal form (NNF).
- We define a "tree" generalization of Σ_n and Π_n formulae, denoted $\mathrm{T}\Sigma_n$ and $\mathrm{T}\Pi_n$ resp., as follows:

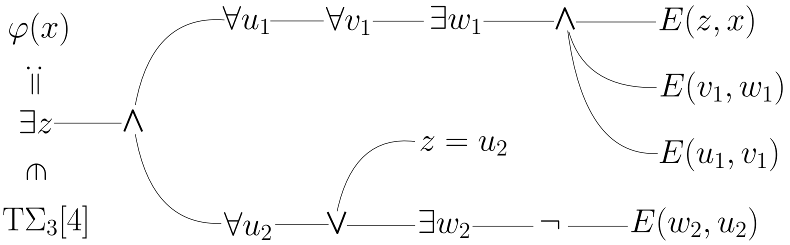
$$\mathrm{T}\Sigma_0 = \mathrm{T}\Pi_0 \quad \Leftrightarrow \quad \text{quantifier-free formulae in NNF}$$

$$\varphi \in \mathrm{T}\Sigma_n \quad \Leftrightarrow \quad \begin{cases} \varphi = \bigwedge \psi_i & \text{where } \psi_i \in \mathrm{T}\Pi_{n-1} \text{ OR} \\ \varphi = \exists x \psi & \text{where } \psi \in \mathrm{T}\Sigma_n \end{cases}$$

$$\varphi \in \mathrm{T}\Pi_n \quad \Leftrightarrow \quad \begin{cases} \varphi = \bigvee \psi_i & \text{where } \psi_i \in \mathrm{T}\Sigma_{n-1} \text{ OR} \\ \varphi = \forall x \psi & \text{where } \psi \in \mathrm{T}\Pi_n \end{cases}$$

- Let $\mathrm{T}\Sigma_n[m]$ and $\mathrm{T}\Pi_n[m]$ resp. denote the subclasses of $\mathrm{T}\Sigma_n$ and $\mathrm{T}\Pi_n$ having formulae of rank at most m.

Example: a $\mathrm{T}\Sigma_3[4]$ formula



$\mathrm{T}\Sigma_0 = \mathrm{T}\Pi_0 \iff$ quantifier-free formulae in NNF

$$\varphi \in \mathrm{T}\Sigma_n \iff \begin{cases} \varphi = \bigwedge \psi_i & \text{where } \psi_i \in \mathrm{T}\Pi_{n-1} \text{ OR} \\ \varphi = \exists x \psi & \text{where } \psi \in \mathrm{T}\Sigma_n \end{cases}$$

$$\varphi \in \mathrm{T}\Pi_n \iff \begin{cases} \varphi = \bigvee \psi_i & \text{where } \psi_i \in \mathrm{T}\Sigma_{n-1} \text{ OR} \\ \varphi = \forall x \psi & \text{where } \psi \in \mathrm{T}\Pi_n \end{cases}$$

Feferman-Vaught decompositions

- Let $\mathcal{L} \in \{\mathrm{T}\Sigma_n[m], \mathrm{T}\Pi_n[m]\}$. Let $\Delta_j = (\psi_{1,j}, \ldots, \psi_{r,j})$ for $j \in \{1, 2\}$ be a sequence of $\mathcal{L}$ sentences.
- For $i \in \{1, \ldots, r\}$ and $j \in \{1, 2\}$, let $X_{i,j}$ be a propositional variable. Let $\mathcal{X}$ be the set of all $X_{i,j}$s, and β be a propositional formula over $\mathcal{X}$.
- The triple $D = (\Delta_1, \Delta_2, \beta)$ is called an $\mathcal{L}$-reduction sequence.
- For disjoint structures $\mathcal{A}_1$ and $\mathcal{A}_2$, we say $(\mathcal{A}_1, \mathcal{A}_2) \vDash D$ if there exists an assignment $\mu : \mathcal{X} \to \{0, 1\}$ such that:

$$\mu \vDash \beta \ \text{ and } \ \mathcal{A}_j \vDash \psi_{i,j} \leftrightarrow \mu(X_{i,j}) = 1 \text{ for } j \in \{1, 2\}$$

- We now say D is a Feferman-Vaught decomposition of an $\mathcal{L}$ sentence φ (over disjoint union), if for disjoint structures $\mathcal{A}_1$ and $\mathcal{A}_2$, it holds that

$$(\mathcal{A}_1 \uplus \mathcal{A}_2) \vDash \varphi \ \leftrightarrow \ (\mathcal{A}_1, \mathcal{A}_2) \vDash D$$

Main results

Theorem

For every $\mathrm{T}\Sigma_n[m]$ ($\mathrm{T}\Pi_n[m]$) sentence φ, there is a $\mathrm{T}\Sigma_n[m]$-reduction sequence ($\mathrm{T}\Pi_n[m]$-reduction sequence) D such that:

1. D is a Feferman-Vaught decomposition of φ.
2. D can be computed from φ in time tower$(n, O((n+1) \cdot |\varphi|^2))$ and the size of D is tower$(n, O((n+1) \cdot |\varphi|))$.

Corollary

Let $\mathcal{L} \in \{\mathrm{T}\Sigma_n, \mathrm{T}\Pi_n\}$. For structures $\mathcal{A}_1$ and $\mathcal{A}_2$, the $\mathcal{L}[m]$ theory of $\mathcal{A}_1 \uplus \mathcal{A}_2$ is determined by the $\mathcal{L}[m]$ theories of $\mathcal{A}_1$ and $\mathcal{A}_2$.

Proposition

Let $\mathcal{L} \in \{\mathrm{T}\Sigma_n, \mathrm{T}\Pi_n\}$ and τ be a vocabulary consisting of predicates of arity $\leq p$. Then upto equivalence, the number of $\mathcal{L}[m]$ formulae $\varphi(\bar{x})$ over τ with $|\bar{x}| = t$, is tower$(n+2, |\tau| \cdot (n+1) \cdot (m+t)^p)$.

Future work

- Various parameterized problems, like k-Vertex cover, k-Clique, k-Dominating Set, belong to $\mathrm{T}\Sigma_n[m]$ with $n = 2$.
- It is known that the model checking problem for FO (also MSO) sentences φ over graphs G of bounded clique-width can be solved in time $f(|\varphi|) \cdot |G|^r$ (indeed with $r = 1$).
- However f above is inherently a non-elementary function of $|\varphi|$ (even for finite trees which have clique-width at most 3).
- The elementary number of formulae in $\mathrm{T}\Sigma_n[m]$ and $\mathrm{T}\Pi_n[m]$ motivates the following question:

Question

For any fixed $k, n \geq 0$, does there exist an algorithm that, given a graph G of clique-width at most k and a $\mathrm{T}\Sigma_n$ or $\mathrm{T}\Pi_n$ sentence φ, decides whether G satisfies φ in time $f(|\varphi|) \cdot |G|^r$ for $r \geq 0$ and f is an elementary function of $|\varphi|$?

|| Dhanyavād! ||

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