PLENITUDE

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Reasoning with *Few* and *Many*

- *Many* and *Few* are indefinite plural descriptions.
- For instance, by “Many apples in the refrigerator” one might mean there are a dozen, whereas by “Many apples in the orchard” one might mean there are 100.
- There are various accounts of reasoning with *Many* in literature. For instance, using syllogisms, generalized quantifiers, etc.
- Our approach is propositional, i.e., we write ‘Many(**a**)’ to indicate that the plural term ‘**a**’ are many. Similarly for *Few*.
- After defining *Few* and *Many,* we prove the completeness theorem for the extended system.

Mid-Plural Logic - Syntax

Plural logic is an extension of first-order logic, in which terms can be of either singular (denoting *an* individual) or plural (denoting individual*s*) sort.

Terms

- Singular (*x, y, z, …*) and Plural (**x**, **y**, **z**, …) Variables
- Constants (c,…)
- $fa_1,a_2,\ldots,a_n$ – where *f* is an n-ary function symbol and $a_1,a_2,\ldots,a_n$ are terms
- *x*:A – where *x* is a variable and A is a formula. This is called exhaustive description
- Plural union (**a,b**), plural intersection (**a.b**), and plural complement (**-a**).

Mid-Plural Logic – Syntax (contd.)

Formulas

- P$\mathbf{a_1}$,$\mathbf{a_2}$,…,$\mathbf{a_n}$ – where P is an n-ary predicate symbol and $\mathbf{a_1}$,$\mathbf{a_2}$,…,$\mathbf{a_n}$ are terms
- **a** $\preccurlyeq$ **b –** where **a** and **b** are terms. Read as **a** is among/included in **b.**
- A ⊃ B, ¬A, …
- ∃*x*A, ∀*x*A

Mid-Plural Logic – Syntax (contd.)

<u>Some Definitions</u>

- E(**a**) $=^{def} \exists x\,(x \preceq \mathbf{a})$ – Says that the term **a** denotes.
- S(**a**) $=^{def} \exists x\,(x = \mathbf{a})$ – Says that the term **a** is singular
- *true* $=^{def} (x = x)$ – Thus *x*:*true* denotes all the individuals
- One can also define the terms plural union (**a,b**), plural intersection (**a.b**), and plural complement (**-a**) using inclusion and exhaustive description.

Mid-Plural Logic – Axiom System

Oliver and Smiley provides a Hilbert-style axiom system. Additional axioms are needed for **:** and ≼. These are –

- $\forall y(y \preccurlyeq x{:}A(x) \equiv A(y))$ - where $A(y)$ has free y wherever $A(x)$ has free x.
- $\mathbf{a} \preccurlyeq \mathbf{b} \equiv E\mathbf{a} \wedge \forall x(x \preccurlyeq \mathbf{a} \supset x \preccurlyeq \mathbf{b})$ - where x is not free in **a** or **b.**
- Sx

Note: One cannot quantify over plurals without sacrificing axiomatizability. This is because allowing quantification over plurals enables one to define a categorical, finitely axiomatized version of Peano Arithmetic.

Mid-Plural Logic – Semantics

- We have individuals (domain)
- A valuation *val* relating objects to terms, relations to predicates, and functions to function symbols, as follows

a. *val x* is an individual and *val* **x** are individuals.

b. *val* P is an n-place relation among individuals, where P is an n-ary predicate.

c. *val f* is an n-place function on individuals, where *f* is an n-ary function symbol.

d. *val x***:**A are the x-variants *val' x* for every x-variant *val'* satisfying A.

The rules for satisfaction are similar to first-order logic.

For inclusion, *val* ⊨ **a**≼**b** *iff* each individual in *val* **a** is in *val* **b** as well.

Introducing *Few* and *Many*

We add three new unary predicate symbols, ‘*Many*’, ‘*Few*’, and ‘$Bet_{Few,Many}$’ to the signature.

$Bet_{Few,Many}$ represents the *penumbra*, i.e., areas of indeterminacy where one cannot commit to either *Few* or *Many.*

Therefore, for each term **a**, we have (new) formulas –

- *Many* (**a**), which says that **a** are many.
- *Few* (**a**), which says that **a** are few.
- $Bet_{Few,Many}$(**a**), which says that **a** are more than few, but not many.

Example Sentences

1. "There are many apples in the refrigerator" – $Many\ (x{:}(\text{apple}(x) \land \text{in.fridge}(x)))$
2. "Every actor has many talents" – $\forall x\ (\text{Actor}\ (x) \supset Many\ (\mathbf{talents}\ (x))$

Note that in (1), "apple" and "in.fridge" are singular predicates.

In (2), **talents** is a multi-valued function. "Actor" is a singular predicate.

Extended Axiom System

To Oliver and Smiley's Hilbert-style system, we add the following axioms for the new symbols.

1. *Many* (*x*:*true*) – Says that the domain has *many* individuals.
2. S**a** ⊃ *Few* **a** – Singletons are few.
3. **a** ≼ **b** ⊃ (*Many* **a** ⊃ *Many* **b**) ∧ (*Few* **b** ⊃ *Few* **a**) – Expresses Anti-monotonicity of *few* and Monotonicity of *many.*

From (3), we can derive *Many* **a.b** ⊃ *Many* **a** and *Many* **a** ⊃ *Many* **a,b**

Extended Axiom System (contd.)

4. *Few* **a** ⊃ ¬ *Many* **a** – *Few* is a degree less than *Many*. Hence partitions plurals into three degrees, which are the new predicates.
5. ¬ *Many* **a** ⊃ *Many* **-a** – Relates negation to *Many*, as well as partitioning *Many* into three degrees, i.e., *Many* **a** ∧ *Many* **–a**, …
6. (i) $\forall x\, (\mathit{Few}\ \mathbf{a} \supset (\mathit{Few}\ \mathbf{a},x \lor \mathit{Bet}_{\mathit{Few,Many}}\mathbf{a},x))$

   (ii) $\forall x\, (\mathit{Bet}_{\mathit{Few,Many}}\mathbf{a} \supset ((\mathit{Bet}_{\mathit{Few,Many}}\mathbf{a},x \lor \mathit{Many}\ \mathbf{a},x) \land (\mathit{Bet}_{\mathit{Few,Many}}\mathbf{a}\text{-}x \lor \mathit{Few}\ \mathbf{a}\text{-}x)))$

   (iii) $\forall x\, (\mathit{Many}\ \mathbf{a} \supset (\mathit{Bet}_{\mathit{Few,Many}}\mathbf{a}\text{-}x \lor \mathit{Many}\ \mathbf{a}\text{-}x))$

   – Sets up degree ordering in the presence of penumbral predicate.

   $\mathit{Few} < \mathit{Bet}_{\mathit{Few,Many}} < \mathit{Many}$

Semantics for the Extension

We provide an intuitive Tarskian semantics for the three degrees introduced.

- *val* ⊨ *Many* **a** *iff* there are many among val(**a**)
- *val* ⊨ *Few* **a** *iff* there are only few among val(**a**)
- *val* ⊨ $Bet_{Few,Many}$ **a** *iff* there are more than few but less than many among val(**a**)

This is justified by noting that *Few, Many* sets up comparison classes.

Thus the first statement says that *val* satisfies *Many* **a** iff there are more than m_a individuals in *val*(**a**), where m_a is indefinite.

Completeness Theorem

- Oliver and Smiley prove the completeness theorem for mid plural logic. They follow Henkin's method of using Lindenbaum construction.
- To prove the contrapositive of completeness theorem, i.e., if *val* $\nvdash$ C then *val* $\nvDash$ C, one starts by adding countable henkin constants *h* to the signature.
- Then, using an enumeration of formulas, construct a maximally consistent set Δ such that Δ $\nvdash$C (for each existential formula for a fresh witness *h* add E(*h*), S(*h*)).
- Use Δ to construct a model with domain consisting of the henkin constants as individuals.

Key Lemma : For all terms **a** and formulas A, *val* **a** are (weakly) identical to **a*** (denoted by *h* s.t. *h* $\preceq$ **a** is in Δ) and *val* satisfies A iff A is one of Δ.

Lindenbaum Construction for the Extended System

When one of *Many* **a**, $Bet_{Few,Many}$**a**, *Few* **a** is encountered in Δ, do the following.

Many **a**

- Find maximal **b** such that $\Delta \vdash \mathbf{b} \preceq \mathbf{a} \wedge \neg$ *Many* **b**
- Add sufficiently many *distinct* $h \preceq \mathbf{a}$, $h \preceq$ **-b**, $\exists x(x = h)$ to Δ if required, to justify *Many* **a** (Plenitude conditions).

$Bet_{Few,Many}$**a**

- Find maximal **b** and minimal **c** such that $\Delta \vdash \mathbf{b} \preceq \mathbf{a} \wedge$ *Few* **b**, $\Delta \vdash \mathbf{a} \preceq \mathbf{c} \wedge$ *Many* **c**
- If required, add henkin constants and preserve separation of counts (as before).

Few **a**

- Find minimal **b** such that $\Delta \vdash \mathbf{b} \preceq \mathbf{a} \wedge \neg$ *Few* **b**
- Preserve separation of counts, and the count must be below distinct $h \preceq$ **b.**

Proof Sketch of Key Lemma for *Many*

Statement :

val satisfies *Many* **a** if and only if the formula *Many* **a** is one of the truth set Δ

Proof Sketch :

- If *Many* **a** is in Δ, by our construction and the plenitude condition, there are many *distinct* henkin constants such that $h \preceq$ **a.**
- For the converse, if ¬ *Many* **a** is in Δ, the construction and the partition axiom ensures that the *distinct* henkin constants such that $h \preceq$ **a** are not many.

Further Work

- Consider the formulas $\forall x\,(A(x,\mathbf{x}) \wedge \mathit{Many}\ \mathbf{x})$ and $(\forall x\, A(x,\mathbf{x}) \wedge \mathit{Many}\ \mathbf{x})$. We might expect the comparison count of **x** for *Many* to depend on the values of *x* in the first case but not the second. But they are equivalent in our system. Can one remedy this, for instance, by employing Skolem functions?
- Incorporating *Most*.
- *Few* and *Many* are vague predicates. Problems of discerning boundary pertain to these predicates as well. We have provided an intuitive Tarskian semantics here. How does one employ this idea of *indefinite descriptions* in the case of, say, the vagueness of Orange-Red?
- Extending the idea of indefinite description to finite (>3) and countably infinite degrees.

THANK YOU