Analytic Multi-Succedent Sequent Calculus for Combining Intuitionistic and Classical Propositional Logic

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Theme of This Talk

- Combining intuitionistic logic (Int) and classical logic (CL):
  - ✓ (C+J) Humberstone 1979, Del Cerro & Herzig 1996
  - ✓ De 2013, De & Omori 2014
  - ✓ Liang & Miller 2013
- Sequent Calculus for C+J

Contribution of This Talk

- Lucio (2000) gave a "structured" sequent calculus for C+J, but we stick to the ordinary notion of sequent.
- Our sequent calculus G(C+J) is cut-free, so the conservativeness of C+J over CL and Int are shown syntactically.
- To get G(C+J), we reveal a "core" of the right rule for intuitionistic implication.

Our Approach

- Our G(C+J) is based on multi-succedent sequent calculus mLJ for intuitionistic logic, proposed by Maehara (1954).
- To add classical implication, the right rule for the intuitionistic implication of mLJ should be reformulated in terms of the context.

Syntax

$$A ::= p \mid \bot \mid A \vee A \mid A \wedge A \mid A \rightarrow_i A \mid A \rightarrow_c A,$$

$$\text{where } p \in \mathsf{Prop}$$

- $\neg_c A$ and $\top$ are abbreviations of $A \rightarrow_c \bot$ and $\bot \rightarrow_c \bot$, respectively.

Kripke Semantics for C+J

by Humberstone 1979.

A model is a tuple $M = (W, R, V)$ where:

- R satisfies reflexivity and transitivity,
- V is a valuation satisfying persistency:

$w \in V(p)$ and wRv jointly imply $v \in V(p)$

- $M, w \models A \rightarrow_i B \Leftrightarrow$
  For all v (wRv & $M, v \models A$ imply $M, v \models B$).
- $M, w \models A \rightarrow_c B \Leftrightarrow M, w \models A$ implies $M, w \models B$.

What Does Simple Addition Cause?

- If the ordinary right rule for $\rightarrow_i$ were used $A \rightarrow_c (B \rightarrow_i A)$ would be derivable:

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow_i B} (\Rightarrow\rightarrow_i)$$

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, B \Rightarrow A} (w \Rightarrow)}{A \Rightarrow B \rightarrow_i A} (\Rightarrow\rightarrow_i)}{\Rightarrow A \rightarrow_c (B \rightarrow_i A)} (\Rightarrow\rightarrow_c)$$

- However, $A \rightarrow_c (B \rightarrow_i A)$ is invalid in the Kripke Semantics!

$\neg_c p \to_c (\mathrm{T} \to_i \neg_c p)$ is invalid

$w \nvDash p$
$w \vDash \neg_c p$
$w \vDash \mathrm{T}$
$w \nvDash \mathrm{T} \to_i \neg_c p$
$w \nvDash \neg_c p \to_c (\mathrm{T} \to_i \neg_c p)$

$v \vDash p$
$v \vDash \mathrm{T}$
$v \nvDash \neg_c p$

- $\neg_c p$ does not satisfy persistency.
- So, persistency cannot be extended to all formulas in C+J.

Hilbert System H(C+J)
by Del Cerro and Herzig 1996

(Per) $A \to_c (B \to_i A)^\dagger$

† any occurrence of $\to_c$ in A is in the scope of $\to_i$

- An underlying idea to get H(C+J) is:
On the top of classical logic, $\to_i$ is added by the idea of conditional logic.

The Right Rule for Intuitionistic Implication

$$\frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow_i B}$$

mLJ* : The same system as mLJ except we replace the right rule of $\rightarrow_i$ with this rule

The ordinary right rule for intuitionistic implication

$\Gamma \Rightarrow \Delta$ is derivable in mLJ
if and only if $\Gamma \Rightarrow \Delta$ is derivable in mLJ*

- We reveal the "core" of the right rule for intuitionistic implication.

Other rules of G(C+J)

- Other rules for intuitionistic connectives are the same as mLJ by Maehara (1954).
- Rules for the classical implication are the ordinary ones:

$$\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow_c B} \ (\Rightarrow \rightarrow_c) \qquad \frac{\Gamma_1 \Rightarrow \Delta_1, A \quad B, \Gamma_2 \Rightarrow \Delta_2}{A \rightarrow_c B, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \ (\rightarrow_c \Rightarrow)$$

Cut-elimination Theorem in $\mathrm{G}(\mathrm{C}+\mathrm{J})$

- We call the combined system $\mathrm{G}(\mathrm{C}+\mathrm{J})$.
- Let $\mathrm{G}^{-}(\mathrm{C}+\mathrm{J})$ be a sequent calculus obtained from $\mathrm{G}(\mathrm{C}+\mathrm{J})$ by removing (Cut).

Cut-elimination Theorem

If $\Gamma \Rightarrow \Delta$ is derivable in $\mathrm{G}(\mathrm{C}+\mathrm{J})$ then $\Gamma \Rightarrow \Delta$ is derivable in $\mathrm{G}^{-}(\mathrm{C}+\mathrm{J})$

- Cut-elimination theorem can be shown by Extended cut rule (Kashima 2009, Ono & Komori 1985):

$$\frac{\Gamma \Rightarrow \Delta, A^m \quad A^n, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \ (ECut)$$

Corollary

$\mathrm{G}(\mathrm{C}+\mathrm{J})$ is a conservative extension of each of intuitionistic and classical logic.

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Further Directions

- We are currently working on decidability and Craig interpolation theorem of $\mathrm{G}(\mathrm{C}+\mathrm{J})$.
- Del Cerro and Herzig (1996) showed the decidability of C+J with the help of translation into S4. We try to give a more direct argument.
- Craig interpolation theorem of $\mathrm{G}(\mathrm{C}+\mathrm{J})$ is not shown straight forwardly .

Thank you!