Polyhedral modal logic

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9th Indian Conference on Logic and its Applications
ICLA

7 March 2021

Spatial logic

- Spatial logic studies various spatial structures though the prism of logic.
- Central to this area is the topological semantics of intuitionistic and modal logics.
- I will discuss a new directions in spatial logic, which I call polyhedral modal logic.
- This topic connects modal logic with polyhedral geometry.

Topological semantics of modal logic

Modal logic

Syntax:

$$\varphi := \perp \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi$$

We will use the shorthand $\Diamond\varphi := \neg\Box\neg\varphi$.

- In Kripke frames $\Box$ and $\Diamond$ are modeled by a binary relation R:

$$\begin{array}{lll} x \models \Box\varphi & \text{iff} & (\forall y)(xRy \Rightarrow y \models \varphi) \\ x \models \Diamond\varphi & \text{iff} & (\exists y)(xRy \text{ and } y \models \varphi) \end{array}$$

- This interpretation is local in that the truth of $\Box\varphi$ and $\Diamond\varphi$ at x is determined by the truth of φ in

$$R[x] = \{y \mid xRy\}$$

- So the truth is determined in the $R[x]$-neighborhood of x.
- This is exactly the idea behind topological semantics of modal logic!

Topological Semantics

- Let X be a topological space, $x \in X$, and U_x an arbitrary open neighborhood of x.

$$
\begin{array}{lll}
x \models \Box\varphi & \text{iff} & (\exists U_x)(\forall y)(y \in U_x \Rightarrow y \models \varphi) \\
x \models \Diamond\varphi & \text{iff} & (\forall U_x)(\exists y)(y \in U_x \text{ and } y \models \varphi)
\end{array}
$$

- The difference between Kripke and topological semantics lies in the fact that while in a Kripke frame $R[x]$ is uniquely determined by x, in a topological space open neighborhoods of x vary, thus yielding more freedom.
- The idea is exactly the same as in the neighborhood semantics of Scott-Montague.

Some history

- One of the first semantics of modal logic is topological, introduced some 20 years before Kripke semantics.
- The pioneers of topological semantics were Tarski (1938), Tsao-Chen (1938), McKinsey (1941), and McKinsey and Tarski (1944).
- They were influenced by the work of Kuratowski (1922) who axiomatized topological spaces by means of closure operators.

Kuratowski's axioms and **S4**

Kuratowski's axioms closely resemble the axioms of Lewis' modal system **S4**:

$\mathbf{c}\varnothing = \varnothing$	$\Diamond\bot \leftrightarrow \bot$
$\mathbf{c}(A \cup B) = \mathbf{c}A \cup \mathbf{c}B$	$\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$
$A \subseteq \mathbf{c}A$	$p \to \Diamond p$
$\mathbf{cc}A \subseteq \mathbf{c}A$	$\Diamond\Diamond p \to \Diamond p$

$\mathbf{i}X = X$	$\Box\top \leftrightarrow \top$
$\mathbf{i}(A \cap B) = \mathbf{i}A \cap \mathbf{i}B$	$\Box(p \wedge q) \leftrightarrow \Box p \wedge \Box q$
$\mathbf{i}A \subseteq A$	$\Box p \to p$
$\mathbf{i}A \subseteq \mathbf{ii}A$	$\Box p \to \Box\Box p$

$\mathbf{S4}$ as the logic of topological spaces

- For a formula φ let $[\![\varphi]\!] = \{x \in X \mid x \models \varphi\}$.
- Then $x \models \Box\varphi$ iff $x \in \mathbf{i}[\![\varphi]\!]$ and $x \models \Diamond\varphi$ iff $x \in \mathbf{c}[\![\varphi]\!]$.
- Thus, $\Box$ is interpreted as interior and $\Diamond$ as closure.
- Consequently, each topological space validates **S4**.
- The converse is also true, and hence **S4** is the logic of all topological spaces when $\Box$ is interpreted as interior and $\Diamond$ as closure.
- But much stronger results hold...

McKinsey-Tarski Theorem

A topological space is dense-in-itself if every point is a limit point.

Theorem

(McKinsey-Tarski, 1944) **S4** is the logic of an arbitrary (nonempty) dense-in-itself metric space.

Remark

The original McKinsey-Tarski result had an additional assumption that the space is separable. In their 1963 book Rasiowa and Sikorski showed that this additional condition can be dropped. Their proof uses the Axiom of Choice.

How to prove the McKinsey-Tarski Theorem

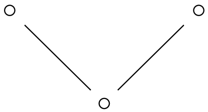
- It is a well-known fact in modal logic that **S4** has the finite model property, meaning that each non-theorem is refuted on a finite Kripke frame, where the binary relation is reflexive and transitive. Such frames are called **S4**-frames.
- Since refuting a formula at a point x of an **S4**-frame $\mathfrak{F}$ only requires the points from $R[x]$, we may assume that $\mathfrak{F}$ is rooted, meaning that there is a point, called a root, such that every point is seen from it.
- Given a dense-in-itself metric space X, the key is to transfer each such finite refutation to X. This can be done by defining an onto map $f : X \to \mathfrak{F}$ that behaves like a p-morphism or functional bisimulation.

How to prove the McKinsey-Tarski Theorem

- Such maps are called interior maps in topology, and they satisfy $\mathbf{i}f^{-1}(A) = f^{-1}(\mathbf{i}A)$ or equivalently $\mathbf{c}f^{-1}(A) = f^{-1}(\mathbf{c}A)$.
- Interior maps are exactly the maps that are continuous (the inverse image of an open set is open) and open (the direct image of an open set is open).
- Constructing such a map from X onto an arbitrary finite rooted **S4**-frame is the main challenge in proving the McKinsey-Tarski theorem.
- But as soon as such a map is constructed, the rest of the proof is easy: each non-theorem φ of **S4** is refuted on a finite rooted **S4**-frame $\mathfrak{F}$. Utilizing $f : X \to \mathfrak{F}$, we can pull the refutation of φ from $\mathfrak{F}$ to X. Thus, each non-theorem of **S4** is refuted on X, yielding completeness of **S4** with respect to X.

Easy example

Let X be the real line $\mathbb{R}$ and $\mathfrak{F}$ the two-fork



Define $f : \mathbb{R} \to \mathfrak{F}$ by sending 0 to the root, the negatives to one maximal node, and the positives to the other maximal node.

How to handle clusters

Define f from $\mathbb{R}$ onto the two-point cluster

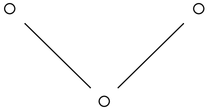
$$\circ \longleftrightarrow \circ$$

by sending the rationals to one node and the irrationals to the other.

More generally, given an n-cluster, partition $\mathbb{R}$ into n-many dense subsets, and send the equivalence classes to the corresponding nodes in the cluster.

The logic of intervals

If we consider the smaller Boolean algebra generated by the open intervals of $\mathbb{R}$, then we can only pick up the two-fork



Theorem (Aiello, van Benthem, G. Bezhanishvili, 2003)

The logic of the two-fork is the logic of the Boolean algebra generated by the open intervals of $\mathbb{R}$.

Euclidean hierarchy

McKinsey and Tarski theorem implies that modal logic of each Euclidean space is **S4**.

However, we can distinguish the logics of Euclidean spaces of different dimensions by restricting the valuation to special subsets.

Theorem (van Benthem, G. Bezhanishvili, Gehrke, 2003)

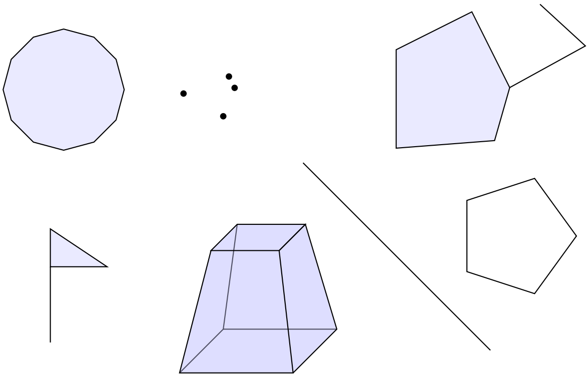
More generally, there is a decreasing sequence of logics $\mathbf{L}_n$ $(n \geqslant 1)$ such that each $\mathbf{L}_n$ is the logic of the Boolean algebra generated by the open hypercubes in $\mathbb{R}^n$. Each $\mathbf{L}_n$ is the logic of the n-product of the two-fork.

This is the beginning of our story...

This is joint work with Sam Adam-Day (Oxford), David Gabelaia (Tbilisi) and Vincenzo Marra (Milan).

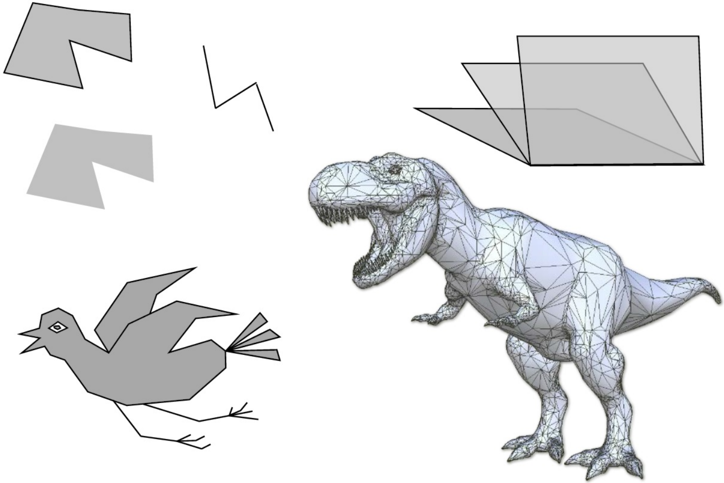
Polyhedral semantics

Polyhedra



- Polyhedra can be of any dimension, and need not be convex nor connected.
- Formally: Boolean combination of convex hulls of finite sets.

Polyhedra



The Boolean algebra Sub(P)

Theorem

The set of subpolyhedra Sub(P) of a polyhedron P forms a Boolean algebra closed under interior and closure.

If one is interested in intuitionistic logic then we have:

Theorem

The set of open subpolyhedra of a polyhedron P is a Heyting algebra.

So we arrive at a polyhedral semantics for modal and intuitionistic logic.

Polyhedral semantcis

Let P be a polyhedron.

A valuation is a map $V : \mathsf{Prop} \to \mathrm{Sub}(P)$.

This valuation is extended to all modal formulas in a standard way:

$$V(\Box\varphi) = \mathbf{i}(V(\varphi)), \quad V(\Diamond\varphi) = \mathbf{c}(V(\varphi)).$$

Then P validates φ (written: $P \models \varphi$) if $V(\varphi) = P$ under any valuation V.

In other words, $P \models \varphi$ if φ is valid in the algebra $\mathrm{Sub}(P)$.

Our aim is to investigate this semantics.

Polyhedral maps

Let P and Q be polyhedra.

A map $f : P \to Q$ is polyhedral if it is a continuous and open, and the inverse image of a subpolyhedron of Q is a subpolyhedron of P.

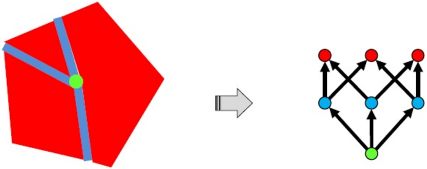
That is, $f^{-1} : \mathrm{Sub}(Q) \to \mathrm{Sub}(P)$ is an algebra homomorphism.

If Q is a finite Kripke frame, then f is polyhedral if it is interior and the inverse image of any point of Q is a subpolyhedron of P.

Polyhedral maps preserve the validity of modal formulas in the polyhedral semantics.

Polyhedral maps

- Connection between polyhedra and posets – open-continuous maps



For polyhedral A and B: If $A \cap B = \emptyset$ and $A \subseteq CB$
Then $\dim(A) < \dim(B)$

Polyhedral semantics

Let P be a polyhedron. Then P validates the Grzegorczyk axiom **grz**, i.e.,

$$P \models \Box(\Box(p \to \Box p) \to p) \to p$$

This is equivalent to saying that a two element cluster is not a polyhedral image of an (open) subpolyhedron of P.

$\mathbf{S4.Grz} = \mathbf{S4} + \mathbf{grz}$ is the logic of finite posets.

If P is a polyhedron of dimension n, then $P \models \mathrm{bd}_{n+1}$ and $P \not\models \mathrm{bd}_{n+2}$, where bd_n is the formula restricting the height of a poset. $\mathbf{BD}_n = \mathbf{S4.Grz} + \mathrm{bd}_n$, for each $n \in \mathbb{N}$.

This is equivalent to saying that an $n+2$-element chain is not a polyhedral image of P.

Thus in the polyhedral semantics we can differentiate Euclidean dimensions.

Gödel embedding

Intuitionistic propositional calculus IPC can be faithfully embedded into **S4.Grz** via the Gödel embedding.

- $Tr(p) = \Box p$,
- $Tr(\varphi \wedge \psi) = Tr(\varphi) \wedge Tr(\psi)$,
- $Tr(\varphi \vee \psi) = Tr(\varphi) \vee Tr(\psi)$,
- $Tr(\varphi \to \psi) = \Box(Tr(\varphi) \to Tr(\psi))$.

Then

$$\mathbf{IPC} \vdash \varphi \text{ iff } \mathbf{S4} \vdash Tr(\varphi)$$

$$\mathbf{IPC} \vdash \varphi \text{ iff } \mathbf{S4.Grz} \vdash Tr(\varphi)$$

S4 is the least modal companion of **IPC** and **S4.Grz** is the greatest modal companion of **IPC**.

All the results in this talk also translate to the intuitionistic setting.

Polyhedral Completeness: Two Approaches

Definition

A logic is polyhedrally complete (poly-complete) if it is the logic of some class of polyhedra.

We investigate the phenomenon of poly-completeness from two directions.

1. Which logics are poly-complete?
2. Given a class of polyhedra, what is its logic?

Local finiteness and FMP

Since $P \not\models \mathrm{bd}_n$ for some $n \in \mathbb{N}$, the algebra $\mathrm{Sub}(P)$ is locally finite.

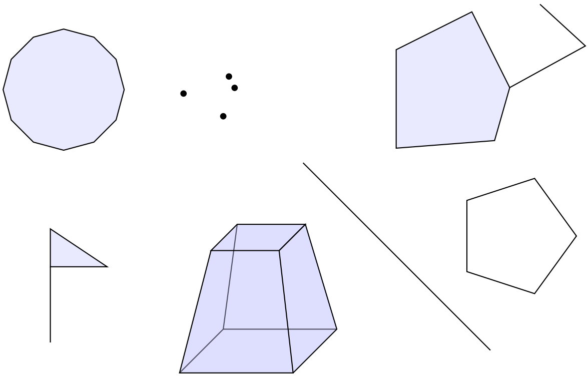
Then $P \not\models \varphi$ implies that there is a finite subalgebra A of $\mathrm{Sub}(P)$ such that $A \not\models \varphi$.

Theorem

Every poly-complete logic has the finite model property (FMP).

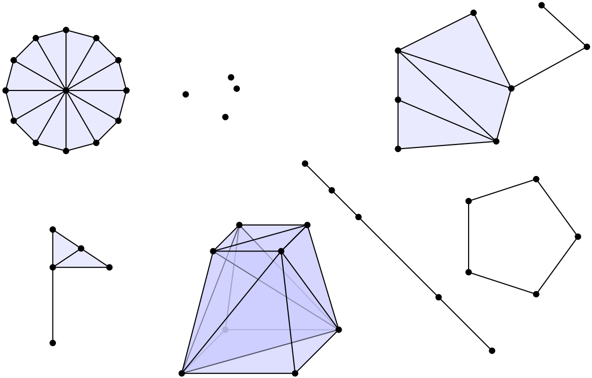
- There exist continuum many modal logics without the FMP. So there exists continuum many poly-incomplete logics.
- From now on we will focus on logics with the FMP.

Polyhedra



- Polyhedra can be of any dimension, and need not be convex nor connected.
- Formally: Boolean combination of convex hulls of finite sets.

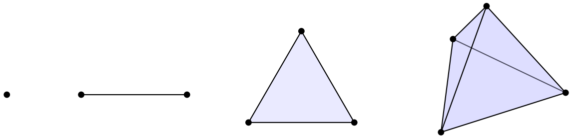
Triangulations I



Intuition: triangulations break polyhedra up into simple shapes.

Triangulations II

- Simplices are the most basic polyhedra of each dimension.
- Points, line segments, triangles, tetrahedra, pentachora, etc.



- A triangulation is a splitting up of a polyhedron into finitely many simplices.
- Represented as a poset $(\Sigma, \preccurlyeq)$ of simplices, where $\sigma \preccurlyeq \tau$ means that σ is a face of τ.
- Its underlying polyhedron is $|\Sigma| := \bigcup \Sigma$.
- Every polyhedron admits a triangulation.

Triangulation Subalgebras

A triangulation is a polyhedral map from P onto Σ.

Definition

Given a triangulation Σ of P, let its triangulation subalgebra $\mathrm{P}(\Sigma)$ be the subalgebra of $\mathrm{Sub}(P)$ generated by Σ.

Lemma

Every finite subalgebra of $\mathrm{Sub}(P)$ is a subalgebra of a finite triangulation subalgebra.

Theorem

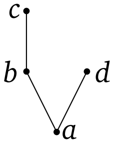
The logic of a polyhedron is the logic of its triangulations.

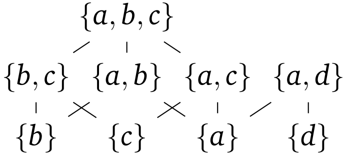
Proof: If $P \not\models \varphi$, then there is a finite subalgebra of $\mathrm{Sub}(P)$ that refutes φ. By the lemma this algebra is a subalgebra of $\mathrm{P}(\Sigma)$ for some Σ. So $\mathrm{P}(\Sigma) \not\models \varphi$.

The Nerve

Definition (Alexandroff's nerve)

The nerve, $\mathcal{N}(F)$, of a finite poset F is the set of all non-empty chains in F, ordered by inclusion.

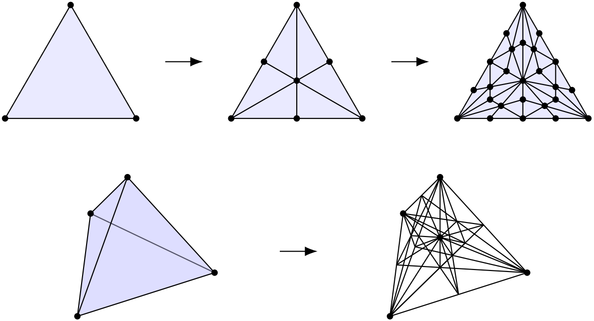




There is always a p-morphism $\mathcal{N}(F) \to F$.

Barycentric Subdivision

Given a triangulation Σ, construct its barycentric subdivision Σ' by putting a new point in the middle of each simplex, and forming a new triangulation around it.



$$\Sigma' \cong \mathcal{N}(\Sigma) \text{ as posets.}$$

Barycentric Subdivision and the Nerve Criterion

Theorem (Nerve Criterion) A logic $\mathcal{L}$ is poly-complete if and only if it is the logic of a class **C** of finite frames closed under $\mathcal{N}$.

- This is about barycentric subdivision.
- Let $\Sigma^{(n)}$ be the nth iterated barycentric subdivision of Σ.
- Intuition: $(\Sigma^{(n)})_{n \in \mathbb{N}}$ captures everything (logical) about $P = |\Sigma|$.
- $\{\mathrm{P}(\Sigma^{(n)}) : n \in \mathbb{N}\}$ approximate $\mathrm{Sub}(P)$.

Consequences

Corollary

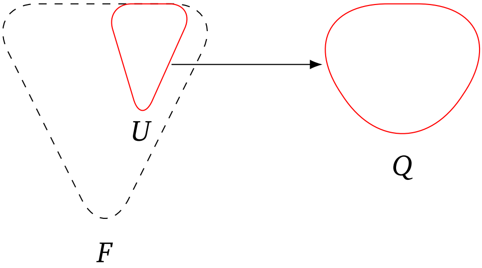
- The logics **S4.Grz** and $\mathbf{BD}_n$ are poly-complete for every $n \in \mathbb{N}$.
- The logics **S4.Grz.2**, **S4.Grz.3**, $\mathbf{S4.Grz.3_n}$, $\mathbf{BW}_n$, $\mathbf{BTW}_n$ and $\mathbf{BC}_n$ are poly-incomplete.
- Moreover, there are continuum-many logics which are poly-incomplete and have the FMP (stable modal logics).

The key idea: (1) use the Nerve Criterion and note that **S4.Grz** is the logic of all finite posets and the nerve construction does not increase the height of a poset.

(2), (3) Note that repeatedly applying $\mathcal{N}$ produces wider and wider frames. Are there other poly-complete logics?

Jankov-Fine Formulas for Forbidden Configurations

For every finite rooted frame Q, there is a formula $\chi(Q)$, the Jankov-Fine formula of Q, such that for any frame F, we have $F \not\models \chi(Q)$ if and only if F up-reduces to Q.

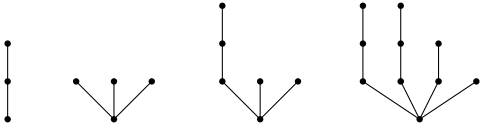


The formula $\chi(Q)$ axiomatizes the least logic which does not have Q as its frame.

Starlike Logics

Definition (starlike tree)

A tree T is starlike if the root is the only branching node.



Definition

A logic $\mathcal{L}$ is starlike if it is of the form
$\mathbf{S4}.\mathbf{Grz} + \chi(T_1) + \chi(T_2) + \cdots$, where $\{T_1, T_2, \ldots\}$ is a (possibly infinite) set of starlike trees other than .

Starlike Poly-completeness I

Theorem (Starlike Poly-completeness)

Every starlike logic $\mathcal{L}$ is poly-complete.

Corollary

$\mathbf{BD}_n + \chi(T_1) + \chi(T_2) + \cdots$ is poly-complete. Hence there are infinitely many poly-complete logics of each finite height.

- Scott's logic $\mathbf{SL} = \mathbf{S4}.\mathbf{Grz} + \chi($$)$.

Corollary

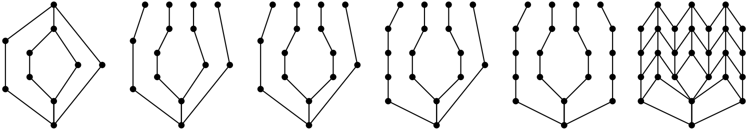
Scott's logic is poly-complete.

Proof of Starlike Poly-Completeness

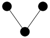
Proof Idea.

- Exploits the Nerve Criterion.
- A method which, given a finite frame F of $\mathcal{L}$, produces a finite frame F' and a p-morphism $F' \to F$ such that $\mathcal{N}^k(F') \models \mathcal{L}$ for every $k \in \mathbb{N}$.
- Two different methods, depending on whether $\chi($$) \in \mathcal{L}$.

□



Starlike Poly-completeness II

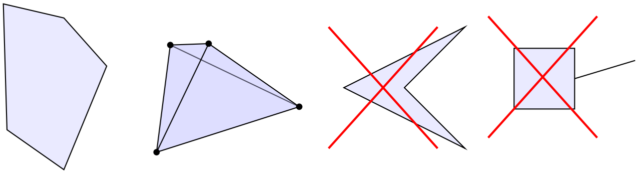
- Starlike logics express quasi-connectedness properties about frames and polyhedra.
- The exclusion of  is necessary: the only poly-complete logic extending **S4**.**Grz** $+ \chi($$)$ is **CPC**.
- The method does not work for arbitrary trees T, and it is unclear if $\chi(T)$ has a sensible geometric meaning.

Convex Polyhedra

We will now look at a different problem: axiomatization.

Definition

A polyhedron P is convex if whenever $x, y \in P$, the straight line from x to y is also in P.

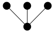


- The most natural class of polyhedra of which to ask: what is its logic?

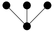
An Axiomatization

Theorem (An Axiomatization)

- The logic of convex polyhedra is axiomatised by

  $\mathbf{S4.Grz} + \chi($$) + \chi($$)$

- The logic of convex polyhedra of dimension n is axiomatised by

  $\mathbf{BD}_n + \chi($$) + \chi($$)$

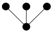
Proof Sketch.

- The **soundness** proof is a combinatorial argument exploiting the Nerve Criterion. Geometric arguments using classical dimension theory are also available.

□

Soundness

That $P \models \chi($$)$ expresses the classical result of Hurewicz and Wallman that a convex polyhedron of dimension n cannot be disconnected by a subset of dimension $< n - 1$.

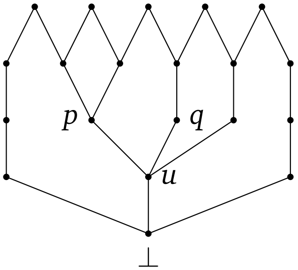
That $P \models \chi($$)$ expresses that a convex polyhedron cannot contain three open disjoint subpolyhedra sharing a common boundary

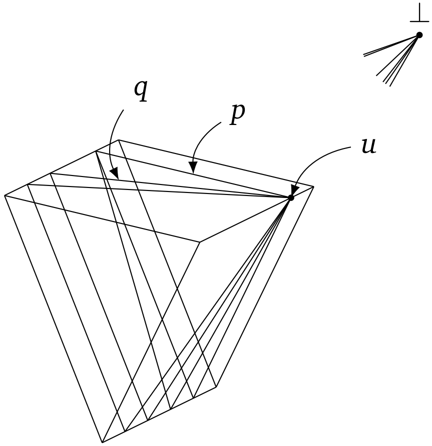
Completeness

Proof Sketch.

- For **completeness**, we show that every finite frame F of the axiomatisation is realised in a convex polyhedron.
- As an intermediary step, transform F into a more geometrically-amenable form, called a saw-topped tree.
- Saw-topped trees are planar, which enables the realisation. □

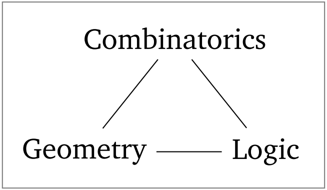
A 4-dimensional Example





Conclusion and future work

We mapped out the following connections.



- Give a full classification of poly-complete modal logics.
- Axiomatize other important classes of polyhedra.
- Path towards applications: polyhedral model checking.

Polyhedral model checking

Spatial model checking is model checking applied to spatial structures and spatial logic.

We would like to develop polyhedral model checker. For example, to reason about 3D images.

The key observation is that the poset obtained by a triangulation keeps all the "logical information" about the polyhedra.

I'll now show a toy prototype (prepared by G.Grilleti and V. Ciancia).

This is joint work with CNR Pisa (Ciancia, Masink, Vallota, Grilletti).

Thank you!