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Panel on logic Education

The place of logic in school curricula and advocacy for logic at school

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An apparent paradox
of the secondary-to-tertiary transition

While the mathematical formalism should allow conceptual clarification, it is observed that it functions for many university students as an almost insurmountable obstacle. This raises the crucial issue of the relationships between logic, language and mathematical reasoning.

Our hypothesis:

The difficulties related to questions of logic and language in the mathematics class are most often insufficiently taken into account by secondary and university teachers.

A common view among teachers is that the learning of mathematics at school is sufficient to develop the logical skills required by advanced mathematics. Hence, an explicit teaching of logical concepts is not needed.

Nevertheless, there are research-based evidence that this is not the case (Durand-Guerrier & al. 2012).

On another side, several authors acknowledge that teaching logic as a separate topic does not necessarily improve mathematical reasoning.

This pleads for the introduction of logical concepts and methods inside the mathematics curriculum by providing activities aiming to make students experience the inter-twinning between mathematical contents and practises and logical skills.

An example

The negation of conditional statement

Negation is a concept that intertwins

Semantics (*exchange of truth values*)*,

Syntax (*construction rules in a given language or in formal languages)*,*

Pragmatic (*interpretation of statements by a subject in a given context*)*.

** These definitions refer to Morris (1938) The theory of signs.*

In upper secondary school in France, the "*counter-example rule"* that a universal statement is false as soon as there is a counter-example to the property at stake is widely taught and used.

However, *negation* is considered as a very simple notion, used from primary school, that does not need to be taught or discussed at this level, and the link between the "*counter-example rule*" and the "*negation of a universal statement*" is not clarified.

The operational form and the predicative form of knowledge.

[…] From a developmental point of view, a concept is all together a set of situations, a set of operational invariants (contained in schemes) and a set of linguistic and symbolic representations.

There are specific characters in mathematical concepts that need to be treated as such.

[…] Finally, the operational form of knowledge and the predicative form are intertwined at all levels. There is no need to oppose one to the other; both are necessary to analyse the difficulties met by children* and the way they can be overcome.

Vergnaud, 2009, p.94.

* *More generally by students*

An activity submitted to undergraduates

1. Give the mathematical negation of the sentence below

« If a number is divisible by 4, then it ends by 4 »
(Si un nombre est divisible par 4, alors il se termine par 4)

2. Say if the sentence below

« If a number is divisible by 4, then it ends by 4 »
(*Si un nombre est divisible par 4, alors il se termine par 4*)

- ✓ Is true
- ✓ Is false
- ✓ is contingent (one can't say)

Justify you answer

The majority of students (among those who answer the question) provide a conditional statement, for example.

- *If an integer is divisible by 4, then it does not end by 4.*
- *If an integer is not divisible by 4, then it does not end by 4.*
- *If an integer is divisible by 4, then it does not necessarily*
- *end by 4.*
- *If an integer is divisible by 4, then it is possible that it does not end by 4.*

The predicative form of knowledge is not available

However, the same students are able to say that the statement

« If a number is divisible by 4, then it ends by 4 »

is false because.

« It is possible to find a number that is divisible by 4 and that does not end by 4 ».

The operational form of knowledge is available

N.B. Some students answer “one can’t say” with the same argument depending on the interpretation they did of “a”. It also happens that some students answer “ true” and justify their answer by providing an example.

The argument

« It is possible to find a number that is divisible by 4 and that does not end by 4 »

operational form knowledge

might be reformulated in the predicative form

« There exits at least a number divisible by 4 and that does not end by 4”.

which is the negation of the initial statement.

However, very few students were able to do it.

Finally, to translate the "*counter-example rule*" in terms of a formal sentence with quantifiers and logical connectors, it is necessary to articulate operational forms of knowledge with predicative forms of knowledge (definition of connectors and quantifiers).

This allows to provide a new predicative form of knowledge:

The negation of a statement in form

"*For all x in U, if P(x), then Q(x)*"

is the statement

« *There exists x in U such that P(x) and not Q(x)* ».

The difficulties to negate an implication might persist until the end of undergraduate studies and beyond.

The connection between the *negation of a universally quantified implication* and the *counter-example rule, that is* widely used in high school is not available for many advanced mathematics students, including students in a Master degree for pre-service mathematics teachers, and even sometimes among in-service secondary mathematics teachers.

What do we learn from such an example

1. Succeeding in mathematical activities at secondary school allows to build “operational invariants” in Vergnaud’s sense, but this is not enough to develop the predicative knowledge needed for advanced mathematics studies.

This highlights the need for specific work in class to articulate both forms of knowledge, which can also be interpreted in term of articulation between syntax and semantics.

2. The rather common practise at university that tends to emphasize the propositional definition of connectors, through truth tables (the classical semantic approach for propositional calculus), and to provide syntactic rules for managing quantified statement (a syntactic approach for predicate calculus), without discussing the interpretation of such rules in mathematical domains miss the core role of the relationships between syntax and and semantics.

This is visible when dealing with these issues in secondary mathematics teacher training. When submitted the same task that the previous one together with the requirement of giving the negation of an explicitly quantified statement studied at university, the great majority of students provide the same answers as undergraduates for the former, but provide a correct answer for the latter by applying the syntactic rules.

A module for PhD students (20h - Workshops)

Initiation to teaching mathematics at University
University of Montpellier, 2016-2019

(Organised by a team of colleagues of the DEMa* Team of the IMAG)

*Didactique et épistémologie des mathématiques

https://imag.edu.umontpellier.fr/equipes-de-recherche/dema/

Program

1. Use student mistakes as leverage for learning
2. Some avenues to promote student engagement in activities offered in tutorials
3. **Work on logical questions with the students (connectors, quantifiers, logical structure of statements, analysis of mathematical proofs)**
4. The teaching of integration in the 1st cycle: what are the objectives? What skills and what concepts need to be developed? What are the links with the teaching of integration in Grade 12 in scientific trail?
5. Tools for thinking about the teaching of linear algebra and promote its learning

Implication:

The maze task (Durand-Guerrier 2003) – raising a disagreement between teachers and students to discuss *implicit quantification* and *letter logical status*

The successor of a prime number to discuss whether a conditional statement with false antecedent should be considered as true:

"Determine the set of integers from 1 to 20 that are such that if n is even, then its successor is prime"

A majority of students, even advanced ones, provide only those numbers that are even and whose successor is prime. Hence, the discussion allows enlightening the distinction between *natural* logic and *mathematical* logic.

Distinction between establishing that an implication is true in a given interpretation (introduction of an implication) and using such implication for making a deduction (elimination of an implication)

Negation

The negation of various sentences, including the examples above, are discussed and some features of negation in French are raised:

- Semantic ambiguities of universal statements in form “All *A* are not *B*” – *linguistic norm (not all A are B)* versus *common use (no A are B)*.

- Default of substitution principle in the linguistic norm: substituting *C* such that *C* is equivalent to *not B*, in “All *A* are not *B*” provides “All *A* are *C*”, which, following the linguistic norm, is not equivalent to “All *A* are not *B*”. For example:

All divisors of 12 are *not even /* All divisors of 12 are *odd*.

This might reinforce the common use, and might introduce misunderstanding specifically for non native speakers (durand-Guerrier & al. 2016).

Analyse of multi-quantified proofs

5 conjectures and potential proofs involving the use of a statement in form “For all x exists y such that $P(x, y)$” are provided to the participants.

Three of them involve two different instances of the universal statement providing two existential statements

There exists y such that P(a, y)

There exists y such that P(b, y).

In the proofs submitted, there is the fallacious step consisting in using the same letter for both existential instantiations, providing the two statements

$P(a, c)$

$P(b, c)$.

Proofs

Analyse of multi-quantified proofs

In one case (*limit of the sum of functions*), the proof was out of a textbook and could be amended to become correct : introducing c_1 and c_2 and considering their minimum.

In an other case (*Cauchy's mean value theorem*), the conjecture is true but the proof is not correct and cannot be amended. The classical proof need the introduction of an auxiliary function.

Both examples are presented and discussed in Durand-Guerrier 2008 p.381-382.

This activity fosters the discussion on the relationships between truth in an interpretation and logical validity, and enlightens the need for teachers to make more rigorous use of the tools supply by logic in their teaching.

One objective of such work on issues of logic, language, reasoning and proof is to enable university (secondary) prospective teachers to develop professional skills allowing them to identify opportunities to work simultaneously on the logical and mathematical aspects of the concepts in play while teaching mathematics .

This supposes in particular taking a critical distance from one's own practice which makes it possible to recognize that certain questions of students put light on unnoticed difficulties referring to questions concerning the articulation between logic, language and mathematical reasoning.

Following Quine (1960), who claims that the *stupidity of our interlocutor is less plausible than a linguistic divergence*, such awareness allows to support the hypothesis of rationality of the students, rather than their inability.

Conclusion

First order logic in the semantic perspective opened by Tarski is a relevant reference for addressing the role of logic in mathematics education, by putting at the core the relationships between syntax and semantics, form and content on the one hand, truth in an interpretation and logical validity in the other hand (Durand-Guerrier 2008, Durand-Guerrier & al. 2012)

The logical analyse of language is a tool to reveal the complexity of mathematical statements and to anticipate the effects on the tasks proposed to students on the one hand (a priori analysis); for analysis of students' work on the other hand (a posteriori analysis).

In Durand-Guerrier, Meyer & Modeste (2019), we advocate that this is also relevant for computer science.

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