Logic for Computer Science
Homework 1

1. Consider an interpretation $(\mathcal{D}, \mathcal{I}, \mathcal{G})$ for a first order language L, and a *sentence* (closed formula) ϕ of L. Show that:

   - $(\mathcal{D}, \mathcal{I}, \mathcal{G}) \models \forall x \phi$ iff $(\mathcal{D}, \mathcal{I}, \mathcal{G}) \models \phi$.
   - $(\mathcal{D}, \mathcal{I}, \mathcal{G}) \models \exists x \phi$ iff there exists $d \in \mathcal{D}$ such that for every G, $(\mathcal{D}, \mathcal{I}, \mathcal{G}_{[x \mapsto d]}) \models \phi$.

   Suppose that ϕ above is not a sentence; what conditions can you place on it for each assertion above to be true ?

2. Define what it means for a term t to be substitutable for a variable x in a formula α.

   - Is x substitutable for y in α if $\alpha : (y = x) \rightarrow \forall y (y = x)$? If so, what is $\alpha[x/y]$?
   - Show that if t is any term and ϕ any sentence, then t is substitutable for any variable x in ϕ.
   - If t is a term and α a formula such that no variable occurs in both t and α then show that t is substitutable for any variable x in α.
   - Prove that for every term t, variable x and formula α, there exists a formula β such that $\models \alpha \equiv \beta$ and t is substitutable for x in β.
   - Prove, for every term t which is substitutable for x in a formula α, an interpretation $(\mathcal{D}, \mathcal{I}, \mathcal{G})$, that $(\mathcal{D}, \mathcal{I}, \mathcal{G}) \models \alpha[t/x]$ iff $(\mathcal{D}, \mathcal{I}, \mathcal{G}_{[x \mapsto (\mathcal{I}, \mathcal{G})(t)]}) \models \alpha$.

3. Which of the following are valid formulas ?

   - $\forall x \exists y.(P(x, y) \rightarrow P(y, y)) \rightarrow \exists y.(P(y, y) \rightarrow P(y, y))$.
   - $\forall x.(\forall x.c = f(x, c) \rightarrow \forall x.\forall x.c = f(x, c))$.
   - $(\exists x.P(x) \rightarrow \exists y \forall z.R(z, f(y))) \rightarrow$ $((\exists x.P(x) \rightarrow \forall y \neg \forall z.R(z, f(y))) \rightarrow \forall x \neg P(x))$.

4. Show that the following sentences of FOL are valid:

   - $\forall x \alpha \rightarrow \forall y (\alpha[y/x])$.

- $\forall x \alpha \rightarrow \exists x \alpha$.
- $\exists x \alpha \rightarrow \forall x \alpha$ if $x \notin FV(\alpha)$.
- $\forall x(\alpha \vee \beta) \rightarrow (\forall x \alpha \vee \beta)$ if $x \notin FV(\beta)$.
- $\forall x(\alpha \rightarrow \beta) \equiv (\alpha \rightarrow \forall x \beta)$ if $x \notin FV(\alpha)$.
- $\forall x(\alpha \rightarrow \beta) \equiv (\exists x \alpha \rightarrow \beta)$ if $x \notin FV(\beta)$.

5. Consider L_O, the first order language of order thory. with one binary predicate symbol $<$, and the structure $(\mathcal{Q}, <_{\mathcal{Q}})$. Which of the following sentences are true in this structure ?

    - $\forall x \exists y.x < y$.
    - $\exists x \forall y(x < y \rightarrow x = y)$.
    - $\forall x \forall y \forall z.(x < y \rightarrow (y < z \rightarrow x < z))$.

6. Consider a first order language with equality and a 2-place predicate symbol LE. Construct a sentence ϕ in this language such that ϕ is true in the structure $(\mathcal{R}, \leq)$ but not in the structure $(\mathcal{N}, \leq)$. (Here $\mathcal{R}$ is the set of real numbers, $\mathcal{N}$ is the set of natural numbers and $\leq$ is the standard ordering on these sets.)

7. In each of the following cases, find a suitable first order language and give axioms in it for the given collections of structures: Sets of size 3; Bipartite graphs; Commutative groups; Fields of characteristic 5.

8. Let L be a finite first order language. Show that, for any *finite* L-structure A, there exists an L-sentence ϕ_A such that all models of ϕ_A are isomorphic to A.

9. Choose a vocabulary and show that there exist two non-isomorphic structures over that vocabulary that are elementarily equivalent structures (i.e. satisfying the same first order sentences).

10. Use the compactness theorem for FOL to show that there is an infinite commutative group in which every element is of order 2. In a similar vein, show that there is an infinite bipartite graph.