Homework 2

Logic for Computer Science

1. Let $\mathcal{L}$ denote the language of classical propositional logic. Let $\Gamma \subseteq \mathcal{L}$ and $\alpha, \beta \in \mathcal{L}$. Show that $\Gamma \cup \{\alpha\} \vdash \beta$ if and only if $\Gamma \vdash \alpha \rightarrow \beta$.

2. Show that the following formulas are tautologies of classical propositional logic:

   (a) $(\alpha \wedge \beta) \leftrightarrow \neg(\neg\alpha \vee \neg\beta)$

   (b) $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \wedge \beta) \rightarrow \gamma)$

3. Show that the following formulas are theorems of classical propositional logic:

   (a) $\neg\neg\alpha \leftrightarrow \alpha$

   (b) $(\neg\alpha \rightarrow \alpha) \rightarrow \alpha$

4. Let us consider the binary number system. When we add two numbers of at most two digits, say ab and cd, we get a number of at most three digits, say pqr. For example, $11 + 10 = 101$. Using standard connectives, write formulas that express p, q, and r as functions of a, b, c, and d.

5. Let $G = (V, E)$ be a graph with a finite non-empty set of vertices, without any self-loops. Let k be a non-zero natural number. For each $x \in V$ and $1 \leq i \leq k$, let $p_{x,i}$ denote a propositional variable. Define a set of formulas, F say, on the set of variables $p_{x,i}$, such that F is satisfiable if and only if G is k-colourable.

6. Given a formula α in classical propositional logic, show that there is a logically equivalent formula β in *conjunctive normal form* (*CNF*). Give an algorithm to compute such an equivalent formula in *CNF*.

7. Consider the following clues to a treasure hunt:

   (a) If this house is next to a lake, then the treasure is not in the kitchen.

   (b) If the tree in the front yard is an elm, then the treasure is in the kitchen.

   (c) The house is next to a lake.

(d) The tree in the front yard is an elm or the treasure is buried under the flagpole.

Where is the treasure? Use semantic consequence relation of Classical Propositional Logic to justify your answer.

8. Find a valuation $V : \{p, q, r\} \to \{0, 1\}$ for each of the following formulas so that the corresponding truth conditions are satisfied:

   (a) $V((p \to (q \to r)) \to ((p \wedge q) \to r)) = 1$

   (b) $V(\neg(\neg p \wedge (p \to q)) \to \neg q) = 1$

   (c) $V(((p \vee q) \to r) \vee ((r \vee q) \to p)) = 0$

9. Without using truth tables, give an algorithm to check whether a formula in the language of Classical Propositional Logic is satisfiable.

10. Let A denote a SAT-solving algorithm, that is, A takes as input a propositional logic formula φ and returns whether or not φ is satisfiable. Using the algorithm A as a subroutine give an algorithm to determine whether two propositional logic formulas φ and ψ are logically equivalent. Prove the correctness of the algorithm.

11. Let M be a maximal consistent set of classical propositional logic formulas. Show that the following holds:

    (a) If $\vdash_{CPL} \alpha$ then $\alpha \in \mathsf{M}$

    (b) $\alpha \vee \beta \in \mathsf{M}$ if and only if ($\alpha \in \mathsf{M}$ or $\beta \in \mathsf{M}$)

12. Consider the following simple 3×3 board to be filled up with the numbers 1, 2 and 3 so that there is no repetition across rows or columns. Fill up the board according to these rules and formalize the reasoning process in a logic.

1	·	·
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