LECTURE 10

15.02.2024

Proof of the claim:

To prove the above claim, it is enough to show the following:

Proposition: Let Γ be a fin-sat and complete set of formulas. Let $\exists x \varphi \in \Gamma$. Let c be a constant symbol not occurring in Γ. Then, $\Gamma \cup \{\varphi[c/x]\}$ is fin-sat.

Proof. Suppose not, that is $\Gamma \cup \{\varphi[c/x]\}$ is not fin-sat. Then is $\Gamma' \subseteq_{fin} \Gamma$, s.t. $\Gamma' \cup \{\varphi[c/x]\}$ is unsatisfiable.

So, $\Gamma' \models \neg\varphi[c/x]$. Now, let $\Gamma' = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$. Then we have,

$\{\gamma_1, \gamma_2, \ldots, \gamma_n\} \models \neg\varphi[c/x]$. Then,

$\models (\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n) \to \neg\varphi[c/x]$ H.W.

Now, e does not occur in $\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n$.

Then $\models (\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n) \to \forall x \neg \varphi$ (Let us assume this for now)

Then, as $\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n \in \Gamma$, we have $\forall x \neg \varphi \in \Gamma$, a contradiction to the fact that $\exists x \varphi \in \Gamma$. Hence we have our result that $\Gamma \cup \{\varphi[e/x]\}$ is fin-sat. This completes the proof. $\square$

Let us now prove the assumption we made above.

Lemma (Generalization of constants)

If $\models \gamma \to \varphi[c/x]$, where c does not occur in γ and φ, then $\models \gamma \to \forall x \varphi$.

Proof. Suppose not. Then, there is a model $\mathcal{M}$, s.t. $\mathcal{M} \not\models \gamma \to \forall x \varphi$. Then, $\mathcal{M} \models \gamma$ and $\mathcal{M} \not\models \forall x \varphi$. Now, $\mathcal{M} \not\models \forall x \varphi$

implies $\mathcal{M}_{[x \to d]} \nvDash \varphi$ for some d in $D_{\mathcal{M}}$.

So, $\mathcal{M}_{[x \to d]} \vDash \neg\varphi$. Now, let us consider a model $\mathcal{M}'$, same as $\mathcal{M}$, but where $I'(c) = d$. Then, $\mathcal{M}'_{[x \to d]} \vDash \neg\varphi$ (as, c does not occur in φ). Similarly, $\mathcal{M}' \vDash \gamma$ (as, $\mathcal{M} \vDash \gamma$ and c does not occur in γ). Then, $\mathcal{M}' \vDash \varphi[c/x]$. So, $\mathcal{M}'_{[x \to I'(c)]} \vDash \varphi$, that is, $\mathcal{M}'_{[x \to d]} \vDash \varphi$.

So, we have a contradiction. Thus, we have $\vDash \gamma \to \forall x \varphi$. This completes the proof. $\square$

Thus, we have shown that Δ is fin-sat, complete and witness-fulfilled, based on earlier proofs and the proof above.

This completes the proof of the proposition. $\square$

Now, we are almost ready to prove our final step: $\underline{\mathcal{M}'_{\Delta} \vDash \forall x \psi \text{ iff } \forall x \psi \in \Delta}$.

We assume Δ to be fin-sat, complete and witness-fulfilled.

Proof of $\mathcal{M}'_\Delta \models \forall x \psi$ iff $\forall x \psi \in \Delta$.

- Suppose $\mathcal{M}'_\Delta \models \forall x \psi$. To show $\forall x \psi \in \Delta$.

  Suppose not. That is, $\forall x \psi \notin \Delta$. Then, $\neg \forall x \psi \in \Delta$. Then, $\exists x \neg \psi \in \Delta$. So, $\neg \psi[t/x] \in \Delta$ for some term t. So, $\psi[t/x] \notin \Delta$. Then, by I.H. $\mathcal{M}'_\Delta \not\models \psi[t/x]$. Then, $\mathcal{M}'_{\Delta[x \to g'_\Delta(t)]} \not\models \psi$. So, $\mathcal{M}'_{\Delta[x \to [t]]} \not\models \psi$. So, $\mathcal{M}'_\Delta \not\models \forall x \psi$, a contradiction.

  So, we have: $\forall x \psi \in \Delta$. This completes the proof. $\square$

- Conversely, suppose that $\forall x \psi \in \Delta$. We need to show that $\mathcal{M}'_\Delta \models \forall x \psi$.

  So, we need to show $\mathcal{M}'_{\Delta[x \to [t]]} \models \psi$ for all terms t. Suppose not. Then,

$\mathcal{M}'_{\Delta[x \to [t]]} \nvDash \psi$ for some term t.

If t is substitutable for x in ψ, we have, $\mathcal{M}'_{\Delta} \nvDash \psi[t/x]$. So, $\mathcal{M}'_{\Delta} \vDash \neg\psi[t/x]$.

Then, by I.H. $\neg\psi[t/x] \in \Delta$. But as, $\forall x \psi \in \Delta$, we have $\psi[t/x] \in \Delta$ (as, $\vDash \forall x \psi \to \psi[t/x]$, where t is substitutable of x in ψ). So, we have a contradiction. Thus, $\mathcal{M}'_{\Delta} \vDash \forall x \psi$.

But here t is substitutable for x in ψ. However, we need the result for all terms t. What do we do?

We prove the following lemma.

<u>Lemma</u>. Given any formula φ, variable x and term t, there exists a formula φ' (an alphabetic variant of φ) obtained by renaming the bound variables in φ, s.t. t is substitutable

for x in φ' and $\models \varphi \leftrightarrow \varphi'$.

H.W. Prove this lemma.

Let us now go back to the converse proof.

Suppose $\forall x \psi \in \Delta$. To prove: $\mathcal{M}'_\Delta \models \forall x \psi$.

Suppose not. Then, $\mathcal{M}'_\Delta \not\models \forall x \psi$. Then,

$\mathcal{M}'_{\Delta[x \to [t]]} \not\models \psi$ for some term t. So,

$\mathcal{M}'_{\Delta[x \to [t]]} \not\models \psi'$ (an alphabetic variant of ψ)

So, $\mathcal{M}'_\Delta \not\models \psi'[t/x]$ (since t is substitutable for x in ψ')

So, $\psi'[t/x] \notin \Delta$ (by I.H.)

So, $\forall x \psi' \notin \Delta$ (since $\models \forall x \varphi \to \varphi[t/x]$, where t is substitutable for x in φ).

So, $\forall x \psi \notin \Delta$, a contradiction.

[H.W. $\models \varphi \leftrightarrow \varphi'$ implies $\models \forall x \varphi \leftrightarrow \forall x \varphi'$]

This completes the proof. $\square$

Thus, we have finished the proof of the statement: for all formulas φ, $\mathcal{M}'_{\Delta} \models \varphi$ iff $\varphi \in \Delta$.

This statement is generally termed as the 'truth lemma'.

Let us recapitulate the proof ideas that we have considered.

1. Start with a fin-sat set Γ
2. Extend the set to form a fin-sat and complete set.
3. Show that the extended set becomes a model set.
4. Then show that this set has a model. This gave the proof of the truth lemma for quantifier-free formulas (zeroth order logic).
5. Extend the fin-sat and complete set

of formulas to a fin-sat, complete and witness-fulfilled set to get the proof of the truth lemma for the quantified formulas.

While doing all these, at some point, we extended the language from $\mathcal{L}$ to $\mathcal{L}'$. Thus the model we constructed is a model for Γ in the language $\mathcal{L}'$. But we need a model for Γ in the language $\mathcal{L}$. Here, $\mathcal{L}: (\mathcal{C}, \mathcal{F}, \mathcal{P})$, and $\mathcal{L}': (\mathcal{C} \cup \mathcal{D}, \mathcal{F}, \mathcal{P})$.

How to get the required model?

Let M' be a model for the language $\mathcal{L}'$, that is $M' = (D', I', g')$. Here:

$$I' \begin{cases} \mathcal{C} \cup \mathcal{D} \\ \mathcal{F} \\ \mathcal{P} \end{cases}$$

Now, we construct a restricted model, $\mathcal{M}'|_{\mathcal{L}}$ as follows: $\mathcal{M} = (D, I, g)$, where

$$D = D', \quad g = g', \quad I \begin{cases} \mathcal{C} \\ \mathcal{F} \\ \mathcal{P} \end{cases} .$$

Now, Γ is a set of formulas in $\mathcal{L}$, so it does not contain any constant symbol from $\mathcal{D}$. In the proof earlier, we constructed a model $\mathcal{M}'_{\Delta}$ for Γ in the language $\mathcal{L}'$. However as no symbol from $\mathcal{D}$ occurs in Γ, $\mathcal{M}'_{\Delta}|_{\mathcal{L}}$ forms a model for Γ as well. And, $\mathcal{M}'_{\Delta}|_{\mathcal{L}}$ is a model w.r.t. the language $\mathcal{L}$. Hence, Γ is satisfiable, that is, every fin-sat set of formulas is satisfiable. This completes the proof of compactness theorem.