Lecture 12

In the last class we finished the proof of completeness theorem for classical propositional logic. We have the axiom system as follows:

Axioms

1. $\varphi \to (\psi \to \varphi)$
2. $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
3. $(\neg\varphi \to \psi) \to ((\neg\varphi \to \neg\psi) \to \varphi)$

Rule

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

Let us go back to the soundness proof before finishing this discussion. To prove soundness, we have to show that these axioms are valid and the rule preserves validity.

Validity of Axiom 1.

Axiom 1: $\varphi \to (\psi \to \varphi)$

To show that Axiom 1 is valid, we need to show that for all valuations V,

$V(\varphi \to (\psi \to \varphi)) = 1$. Suppose not. Then there is a valuation, V', say, s.t. $V'(\varphi \to (\psi \to \varphi)) = 0$

Then, $V'(\varphi) = 1$ and $V'(\psi \to \varphi) = 0$. Now, $V'(\psi \to \varphi) = 0$ implies $V'(\psi) = 1$ and $V'(\varphi) = 0$.

So, we arrive at a contradiction.

Hence, the result.

H.W. Prove that Axioms 2 and 3 are valid.

Rule preserves consequence / validity.

$$\frac{\varphi \quad \varphi \to \psi}{\psi}$$

We need to show that, for any valuation V, if $V(\varphi) = 1$ and $V(\varphi \to \psi) = 1$, then $V(\psi) = 1$.

Now, let V' be a valuation s.t. $V'(\varphi) = 1$ and $V'(\varphi \to \psi) = 1$. Then, of course, $V'(\psi) = 1$.

This completes the proof.

So, finally we have the soundness result of Classical Propositional Logic. Thus we have : $\Gamma \vdash \varphi$ iff $\Gamma \vDash \varphi$.

This result is termed as the general completeness result. We also have a weak completeness result for CPL:

$$\vdash \varphi \quad \text{iff} \quad \vDash \varphi$$

φ is a theorem $\qquad$ φ is a validity.

H.W. Prove the weak completeness result for CPL, independent of the general result.

Theorems in CPL.

1. $\vdash \varphi \to \varphi$.

Proof (a) $\{\varphi\} \vdash \varphi$.

$$\vdash \varphi \to \varphi \quad (\text{by D.T.}).$$

Proof (b) 1. $\varphi \to ((\varphi \to \varphi) \to \varphi)$ Axiom 1.

2. $(\varphi \to ((\varphi \to \varphi) \to \varphi)) \to ((\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi))$

Axiom 2.

3. $(\varphi \to (\varphi \to \varphi)) \to (\varphi \to \varphi)$ (M.P. 1,2)

4. $\varphi \to (\varphi \to \varphi)$ Axiom 1.

5. $\varphi \to \varphi$ (M.P. 4,3).

2. $\vdash \neg\neg\varphi \to \varphi$.

Proof

1. $\neg\neg\varphi \to (\neg\varphi \to \neg\neg\varphi)$ (Axiom 1)
2. $\{\neg\neg\varphi\} \vdash \neg\varphi \to \neg\neg\varphi$ (D.T.)
3. $\neg\varphi \to \neg\varphi$ (Theorem)
4. $(\neg\varphi \to \neg\varphi) \to ((\neg\varphi \to \neg\neg\varphi) \to \varphi)$ (Axiom 3)
5. $(\neg\varphi \to \neg\neg\varphi) \to \varphi$ (M.P. 3,4)
6. φ (M.P. 2, 5)
7. $\neg\neg\varphi \to \varphi$ (D.T.).

H.W.

1. $\vdash_{CPL} \varphi \to \neg\neg\varphi$
2. $\vdash_{CPL} \varphi \to (\varphi \vee \psi)$
3. $\vdash_{CPL} \varphi \to (\psi \to (\varphi \wedge \psi))$