LECTURE 13

06.03.2024

We have shown earlier that isomorphic structures together with corresponding assignment functions that are given in a certain way, satisfy the same first-order formulas. Let us now show another corollary of this result.

Proposition: Let L be a first-order language and $\mathcal{A}$ be an L-structure. Let R be an n-ary relation on $D_{\mathcal{A}}$, such that R is definable in L. Let h be an automorphism on $\mathcal{A}$. Then:

$$(a_1, a_2, \ldots, a_n) \in R$$
iff
$$(h(a_1), h(a_2), \ldots, h(a_n)) \in R.$$

Proof: Let $\varphi(x_1, x_2, \ldots, x_n)$ be a formula that defines R in $\mathcal{A}$. Then, we have:

$$(a_1, a_2, \ldots, a_n) \in R$$

iff $\mathcal{A}_{[x_1 \to a_1,\, x_2 \to a_2,\, \ldots,\, x_n \to a_n]} \models \varphi$

iff $\mathcal{A}_{[x_1 \to h(a_1), x_2 \to h(a_2), \ldots, x_n \to h(a_n)]} \models \varphi$.

iff $(h(a_1), h(a_2), \ldots, h(a_n)) \in R$.

This completes the proof.

H.W. Using this result show that $\{b\}$ is not definable in the example on graphs given above.

Another example.

Consider the structure $(\mathbb{R}, <)$. Now, $\mathbb{N} \subseteq \mathbb{R}$. We show that $\mathbb{N}$ is not definable in $\mathbb{R}$, given the language $\mathcal{L}$ having a binary predicate symbol, P, say, whose interpretation in $\mathbb{R}$ is given by $<$. How to prove this?

We show this by using an automorphism on $\mathbb{R}$: $h(x) = x^3$. Since h is an automorphism, if $\mathbb{N}$ had been definable, we would have:

$$n \in \mathbb{N} \text{ iff } h(n) \in \mathbb{N}$$

But, there are elements outside $\mathbb{N}$, which get mapped in $\mathbb{N}$. Thus, $\mathbb{N}$ cannot be definable in $(\mathbb{R}, <)$.

Till now we were discussing different aspects of satisfiability of first-order formulas in first-order models and also the semantic consequence relation $\models$. By compactness theorem we have that: if $\Gamma \models \varphi$, then there is $\Gamma_0 \subseteq_{fin} \Gamma$, such that $\Gamma_0 \models \varphi$.

- What is this Γ_0? Can we have some way to generate this Γ_0?
- What about the kind of mathematical reasoning that we do while trying to prove various results?

Deductive consequence relation

$\Gamma \vdash \varphi$: φ is a deductive consequence of Γ.

Q. What is the relationship between two consequence relations?

The following results give us our answer:

Soundness Theorem.

If $\Gamma \vdash \varphi$ then $\Gamma \vDash \varphi$.

Completeness Theorem.

If $\Gamma \vDash \varphi$ then $\Gamma \vdash \varphi$.

Deductive consequence relation can be defined in various ways, e.g., Hilbert-style axiomatization, Gentzen's sequent calculus, and Gentzen's natural deduction.

Definition of $\Gamma \vdash \varphi$:

Let Γ be a set of formulas and φ be a formula. φ is said to be a deductive consequence of Γ $(\Gamma \vdash \varphi)$ if there is a finite sequence of formulas $\varphi_1, \varphi_2, \ldots, \varphi_n$, s.t.

- φ_n is φ
- each φ_i is either a <u>member of Γ</u> or an <u>axiom</u> or obtained by the application of some <u>rule of inference</u>.

What are these axioms and rules?

- Axioms are formulas in the language.
- A rule of inference is a subset of $\wp(\mathcal{L}) \times \mathcal{L}$, where $\mathcal{L}$ denotes the language under consideration.

We write them as : $\dfrac{\psi_1, \psi_2, \ldots, \psi_k}{\psi}$ - premise

- consequence

Soundness Theorem

If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$.

Q. What kind of properties would we like to prove for these axioms and rules to get our soundness theorem?

Let us explore - - - .

We have: $\Gamma \vdash \varphi$. So, we have:

$\Gamma \vdash \varphi_1$ . . To prove: $\Gamma \models \varphi_1$

$\Gamma \vdash \varphi_2$ To prove: $\Gamma \models \varphi_2$

$\vdots$ $\vdots$

$\Gamma \vdash \varphi_n (= \varphi)$ To prove: $\Gamma \models \varphi_n = \varphi$

- We need to apply induction on the length of this finite sequence.

Notation:

The sequence $\varphi_1, \varphi_2, \dots, \varphi_n = \varphi$ which we mentioned in the definition of $\Gamma \vdash \varphi$ is termed as a proof of φ from Γ.

Thus, from above, we basically need to apply induction on the length of proof of φ from Γ.

For the base case, where $n=1$,

$\varphi_n \in \Gamma$ or φ_n is an axiom.

- if $\varphi_n \in \Gamma$, then of course $\Gamma \models \varphi_n$.
- if φ_n is an axiom, we would have $\Gamma \models \varphi_n$, if we can show that φ_n is a validity.

* We need to check that all axioms are validities

For the induction step, where $n = m+1$ for some $m \geq 1$; φ_n can be

(i) a member of Γ

(ii) an axiom

(iii) obtained by some rule.

* To take care of case (iii), we need to show that if any model satisfies the premises of a rule it will also satisfy the consequence of the rule as well. In other words, rules preserve the consequence relation.

Thus to prove soundness, we show:

(1) Axioms are validities.

(2) Rules preserve consequence

The proof then follows!

Proving Soundness Theorem :

We prove by applying induction on the length of a proof of φ from Γ.

Base Case : $n = 1$

Then φ is either an axiom or a member of Γ. Then we have that $\Gamma \models \varphi$ [if φ is a member of Γ, then any model of Γ would satisfy φ, and if φ is an axiom, then φ is a validity (see (1) above)].

Induction Hypothesis $n \leq m$

The result holds when the length of a proof of φ from Γ is $\leq m$.

Induction Step $n = m+1$

Then φ is either an axiom

or a member of Γ or obtained by some rule of inference. The first two cases have already been dealt with in the base case.

Now, suppose φ is obtained by some rule of the form $\dfrac{\psi_1, \psi_2, \ldots \psi_k}{\psi (= \varphi)}$.

Then, $\psi_1, \psi_2, \ldots, \psi_k$ have occured in the proof of φ from Γ. So, by I.H., we have $\Gamma \models \psi_1$, $\Gamma \models \psi_2$, $\ldots$, $\Gamma \models \psi_k$. Now, since rules preserve consequence, we have $\Gamma \models \psi$, that is, $\Gamma \models \varphi$. This completes the proof.